Middlemen, Inventories and Economic Dynamics*

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Abstract

We study dynamics in frictional markets with intermediated trade, where middlemen buy goods or assets from sellers, hold them as inventory, and sell when contacting appropriate buyers. Buyers in the model have heterogeneous match-specific valuations, letting us characterize equilibrium in terms of reservation trading strategies (related papers with homogeneous valuations imply bang-bang solutions that are awkward for the economics and mathematics). Using bifurcation theory, we show there are equilibria where market participation, volume, prices, liquidity and other variables fluctuate as self-fulfilling prophecies. The dynamics emerge from strategic considerations, not mechanical assumptions, like increasing returns or related devices in related models.

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1 Introduction

This paper studies economic dynamics in frictional markets with intermediated trade. As in related papers, in our framework middlemen buy goods or assets from sellers, hold them in inventory, and sell when they contact appropriate buyers. A key extension over past work is that match-specific valuations of buyers are heterogeneous, allowing a characterization of outcomes in terms of reservation trading strategies. This is useful because models with homogeneous valuations imply bang-bang (corner) solutions that create difficulties for the economics and mathematics. We prove there exist multiple equilibria, including cycles in continuous and in discrete time, where market participation, trading volume, prices, liquidity and other variables fluctuate as self-fulfilling prophecies. The analysis makes use of bifurcation methods that are not applicable with bang-bang solutions.

Our approach builds on research, going back to Rubinstein and Wolinsky (1987), using search-and-bargaining theory to show how roles for intermediaries can emerge from their advantages in certain attributes, including search efficiency, bargaining power, information, storage costs, inventory capacity, or the ability to use credit.[1] Most of these papers concentrate on steady state, or sometimes on transitions to steady state, while we emphasize endogenous – i.e., belief-based – fluctuations. While some previous studies hint at the possibility of endogenous fluctuations in intermediated markets, they do not prove that such equilibria exist.

The objects being traded can be either assets or goods, the difference being that inventories of assets yield positive returns, while goods yield negative returns one can interpret as storage costs. That distinction is taken from Nosal et al. (2019), a model that is related to ours, but different in important ways explained in Section

[1] There are dozens of papers worthy of citation, but to save space we refer readers to an online bibliography at https://github.com/qiao-ziqi/middlemen. Here are just a few examples: in Rubinstein and Wolinsky (1987) middlemen meet buyers faster than sellers meet buyers; in Biglasiser (1993) and Li (1998) they have informational advantages; in Masters (2008) and Nosal et al. (2015) they have higher bargaining power or lower costs; in Shevchenko (2004) and Watanabe (2010) they hold more inventory; in Gong et al. (2025) they are better at using credit. Of these papers, only Gong et al. (2025) study endogenous fluctuations, but their mechanism is entirely different, relying on the well-know feature that imperfect credit models can display multiplicity and belief-based dynamics, which is not relevant here.

4.5. In any case, having agents trading assets connects us to a literature on dealers in OTC (over the counter) financial markets following Duffie et al. (2005), recently surveyed by Hugonnier et al. (2024). However, inventories – which play a big role here – are absent in those models: dealers simply transfer assets between sellers and buyers using a frictionless interdealer market (with exceptions, e.g., Weill 2008, but he does not consider belief-based dynamics).

Our dynamics emerge from the interaction of inventories, heterogenous valuations, and endogenous entry by sellers. When middlemen meet buyers, they could trade, but there is the option of holding out for a higher-valuation buyer. Agents use reservation strategies, as in standard search theory, but now there is a strategic effect. Namely, if reservation values are low middlemen are more inclined to sell to buyers, and hence more often need to replenish inventories. That makes it easier for sellers to trade, increasing seller entry and making it easier for middlemen to replenish inventories, thus rationalizing low reservation values. And if reservation values are high middlemen are less likely to sell and less often need to replenish inventories, thus lowering seller entry, making it harder to replenish inventories and rationalizing high reservation values. This strategic complementarity can lead to multiplicity and cycles.

Interpreting the framework in terms of retail goods markets has some interesting implications. For one, a stylized fact is that the efficiency or productivity of these markets differs dramatically across economies, as discussed by Lagakos (2016). Multiplicity is consistent with the idea that retail markets in some economies may be stuck in a bad equilibrium, where low efficiency or productivity is a self-fulfilling prophecy.[2] A different motivation comes from the idea that intermediation is related to volatility. Gehrig and Ritzberger (2022) push this and provide references to empirical work supporting it. The basic idea is that middlemen speed up trade, which may be desirable, but can also increase volatility. We agree that the relation-

[2] Given his expertise, we quote Lagakos, with permission, from correspondence: "That does sound intriguing – I don't remember seeing a paper that says something like that. I have the impression that even countries of similar income levels often have pretty different retail structure and efficiency. That smells like it could be multiple equilibria."

ship between intermediation and volatility is worth investigation, and think that dynamic search theory is an attractive way to analyze this.

In particular, we show using bifurcation theory the existence of limit cycles. The use of these methods goes back to Benhabib and Nishimura (1979) in growth theory. Applications in search include Diamond and Fudenberg (1989), who get cycles in Diamond (1982a) *if* the matching technology displays increasing returns, and Mortensen (1999), who gets cycles in a version of Pissarides (2000) *if* the production technology displays increasing returns. Whatever the empirical relevance of increasing returns, it seems fair to say those results are driven by mechanical technology specifications that play no role here.[3]

There are also papers with multiplicity and endogenous dynamics in monetary economics (see Rocheteau and Wright 2013 and references therein). The forces behind those results are different, relying on the notion that what you accept in exchange depends on what others accept. In particular, as explained above, our results work through inventories, heterogenous valuations and endogenous entry, factors that are not at all necessary in monetary theory.[4] One feature of monetary models is that utility is not perfectly transferable. Burdett and Wright (1998) show even nonmonetary search models with nontransferable utility can have multiplicity and dynamics while otherwise similar models with transferable utility cannot (see also Martel et al. 2023). This plays not role here, as we have transferable utility.

Motivating a general interest in inventories, there is much discussion of how they are an important component of business cycles (Blinder 1990 is a classic, while Khan and Thomas 2007 is a more recent, example). This is at least partly because they are volatile and procyclical, as can be understood with a supply-side story: when productivity is high, it is efficient to produce a lot and keep some

[3]Other search models with dynamics based on increasing returns or related devices include Howitt and McAfee (1988,1992), Boldrin et al. (1993), Kaplan and Menzio (2016) and Sniekers (2018). Fershtman and Fishman (1992), Burdett and Coles (1998) and Albrecht et al. (2013) are examples of search models with somewhat different dynamics.

[4]We mention that early work by Kehoe et al. (1993) and Renero (1988) study cycles in the discrete-time model of Kiyotaki and Wright (1989), but Oberfield and Trachter (2012) show the cycles vanish as period length shrinks. This is a reason to consider both continuous and discrete time, as discussed in detail in Section 4.3.

as inventory to spread good times into the future. Ours is a demand-side story: holding productivity constant, when inventories are high, production slows because middlemen are not buying. This could make inventories countercyclical if there were no other shocks, but of course there are other shocks. In any case, this paper is not trying to account for the macro data, it is a micro model of what can happen as a self-fulfilling prophecy about inventories, trading strategies and entry.

In what follows, Section 2 presents a simple specification, without entry, where there is a unique equilibrium. Section 3 adds entry by sellers and shows how multiplicity and cycles emerge. Section 4 explores other topics, including welfare, entry by middlemen, discrete-time models, and a setup where consumers instead of middlemen hold inventories. Section 5 concludes.

2 The Basic Framework

A continuum of infinitely-lived, risk-neutral agents come in three types, labeled B, S and M, for buyers, sellers and middlemen. Type i can participate in a continuous-time, bilateral matching market if they pay entry cost κ_i, but for now $\kappa_i = 0 \; \forall i$ so everyone participates. Indeed, they participate forever, which is not crucial but simplifies some calculations compared to, e.g., Rubinstein and Wolinsky (1987) where M stays forever while B and S exit after one trade (see also Nosal et al. 2015, Vayanos and Wang 2007 and Farboodi et al. 2023). When B and S meet, S can produce an indivisible object x at 0 cost that gives B match-specific payoff π, with CDF $F(\pi)$ on $[\underline{\pi}, \overline{\pi}]$. Note that π can be utility if B consumes x, or profit if B uses it as an asset for investment or input for production.

Also, S could produce for M, who can store x in inventory and may or may not sell it to B when they meet. While M can store x, for now neither S nor B want to store it – S prefers to produce just before trade and B prefers to consume right after trade – but that changes in Section 4.4. There is a flow payoff ρ for M with x in inventory, and we say x is an asset if $\rho > 0$ (it has a return), while x is a good if $\rho < 0$ (it has a storage cost), usage that is not especially important but helps keep track of cases. Holdings of x by M are constrained to $\{0,1\}$, which is special, but allows one to make salient points in a succinct way in many applications of search theory.[5] Inventories held by M depreciate by disappearing at rate $\delta \geq 0$.

Let n_i be the measure of $i = B, S, M$ in the market, which is exogenous for now, and n the measure of M with x in inventory, which is endogenous. There is a standard matching technology: you meet someone at Poisson rate α; and each meeting is a random draw from the population. In particular, if N is the measure of all market participants, the arrival rate of B for both M and S is $\alpha n_b/N$, so M has no advantage over S in search. Given $\pi > 0$, when B and S meet they always trade, since it does not affect continuation values. When S meets M with $x = 0$ they trade unless $\rho < 0$ and $|\rho|$ is big (more on this below). The question is, when M with $x = 1$ meets B, do they trade? As will be shown, the answer depends on fundamentals, including π, of course, but also on beliefs.

If j gives x to i, the latter pays p_{ij} determined by bargaining with transferable utility. Thus, if Σ_{ij} is the total surplus available when i and j meet, they trade if $\Sigma_{ij} > 0$, and i's surplus is $\theta_{ij}\Sigma_{ij}$, where $\theta_{ij} \geq 0$ is i's bargaining power against j, with $\theta_{ij} + \theta_{ji} = 1$. Letting $V_{s,t}$ and $V_{b,t}$ be the value functions for S and B, $V_{x,t}$ the value function for M with $x \in \{0,1\}$, and $\Delta_t = V_{1,t} - V_{0,t}$, we have[6]

$$\Sigma_{bs,t} = \pi,\ \Sigma_{ms,t} = \Delta_t,\ \text{and}\ \Sigma_{bm,t} = \pi - \Delta_t. \tag{1}$$

Note the continuation values and threat points for S and B cancel in the surpluses, so V_s and V_b do not appear. From these follow what we call the direct price, the wholesale price, and the retail price, given respectively by

$$p_{bs,t} = \theta_{sb}\pi,\ p_{ms,t} = \theta_{sm}\Delta_t,\ \text{and}\ p_{bm,t} = \theta_{mb}\pi + \theta_{bm}\Delta_t. \tag{2}$$

[5] In addition to middlemen models à la Rubinstein and Wolinsky (1987), examples with $\{0,1\}$ restrictions include the original search-equilibrium model of Diamond (1982a), the monetary models cited in fn. 4 and many others, banking models like Cavalcanti and Wallace (1999), OTC asset models like Duffie et al. (2005), labor models like Pissarides (2000) and partnership models like Burdett and Coles (1997). One can allow inventories in, say, $\{0,1,2,...\}$, but then one must use numerical methods, and we are after analytic results.

[6] At this point we start subscripting variables by t, including n_s, N and n_m even though they are constant in this most basic version of the environment, so that the same expressions hold when they are endogenous; we do not subscript n_b by t since it is fixed in all versions.

When M with $x = 1$ meets B and the match-specific valuation is π, they trade with probability $\tau_t = \tau(\pi, R_t)$, where R_t is the reservation value:

$$\tau(\pi, R_t) = \begin{cases} 0 & \text{if } \pi < R_t \\ [0,1] & \text{if } \pi = R_t \\ 1 & \text{if } \pi > R_t \end{cases} \tag{3}$$

Clearly, $R_t = \Delta_t = V_{1,t} - V_{0,t}$. Hence, the expected flow payoff for B is

$$rV_{b,t} = \frac{\alpha n_{s,t}}{N_t}\theta_{bs}\mathbb{E}\pi + \frac{\alpha n_t}{N_t}\theta_{bm}\mathbb{E}\left[\tau(\pi, \Delta_t)(\pi - \Delta_t)\right] + \dot{V}_{b,t}, \tag{4}$$

where r is the discount rate and and prices have been eliminated using (2). The first term on the RHS is the arrival rate of S times B's share of the surplus; the second is the arrival rate of M with $x = 1$ times the probability they trade times B's share of the surplus; the third is the pure time change in value.

Similarly, for S,

$$rV_{s,t} = \frac{\alpha n_b}{N_t}\theta_{sb}\mathbb{E}\pi + \frac{\alpha(n_{m,t} - n_t)}{N_t}\theta_{sm}\Delta_t + \dot{V}_{s,t}, \tag{5}$$

and for M,

$$rV_{0,t} = \frac{\alpha n_{s,t}}{N_t}\theta_{ms}\Delta_t + \dot{V}_{0,t} \tag{6}$$

$$rV_{1,t} = \frac{\alpha n_b}{N_t}\theta_{mb}\int_{\Delta_t}^{\infty}(\pi - \Delta_t)\, dF(\pi) + \rho - \delta\Delta_t + \dot{V}_{1,t}. \tag{7}$$

Subtracting (7)-(6) and simplifying, using integration by parts, we get

$$\dot{\Delta}_t = -\frac{\alpha n_b}{N_t}\theta_{mb}\int_{\Delta_t}^{\infty}[1 - F(\pi)]\, d\pi + \frac{\alpha n_{s,t}}{N_t}\theta_{ms}\Delta_t - \rho + (r + \delta)\Delta_t. \tag{8}$$

The evolution of inventories held by M is

$$\dot{n}_t = \frac{\alpha n_{s,t}(n_{m,t} - n_t)}{N_t} - \frac{\alpha n_b n_t \mathbb{E}\tau(\pi, \Delta_t)}{N_t} - \delta n_t, \tag{9}$$

where $\mathbb{E}\tau(\pi, \Delta) = \Pr(\pi > \Delta)$ is the unconditional probability that M and B trade. The first term on the RHS is the measure of M without x times the rate at which they buy it from S; the second is the measure of M with x times the rate at which they sell it to B; the third is depreciation.

Equilibrium is defined as a nonnegative and bounded path for (Δ_t, n_t) satisfying the dynamical system (8)-(9) with initial condition n_0 giving inventories at $t = 0$.[7] A steady state is a constant (Δ, n) satisfying (8)-(9). Given an equilibrium (Δ_t, n_t), or steady state (Δ, n), all other variables follow easily, including payoffs, prices, trade volume, etc.

With no intermediaries, $n_m = 0$, equilibrium is obviously unique, with B and S trading whenever they meet. With $n_m > 0$, first notice that the path of Δ_t is independent of n_t. Then from (8)

$$\frac{\partial \dot{\Delta}_t}{\partial \Delta_t} = \frac{\alpha n_b}{N_t}\theta_{mb}\left[1 - F\left(\Delta_t\right)\right] + \frac{\alpha n_{s,t}}{N_t}\theta_{ms} + r + \delta > 0,$$

implying Δ_t must equal its steady state value $\forall t$, since any other solution to (8) diverges – a result that reappears in some, but not all, formulations below, and is discussed more later. Given Δ, (9) implies

$$\frac{\partial \dot{n}_t}{\partial n_t} = -\left[\frac{\alpha n_{s,t}}{N_t} + \frac{n_b \mathbb{E}\tau\left(\pi, \Delta\right)}{N_t} + \delta\right] < 0,$$

so n_t converges monotonically to its steady state. This proves:

Proposition 1 *Without entry equilibrium is unique: there are no dynamics due to self-fulfilling expectations.*

3 The Main Model

Now let S face a participation decision, which is natural and nice because it lets us compare economies with and without middlemen while keeping the environment otherwise the same.[8] Then $n_{s,t}$ and N_t can vary with time, while n_m and n_b are constant. The entry condition $rV_{s,t} = \kappa_s$ implies $\dot{V}_{s,t} = 0$. Then from (5) we get

$$N_t \kappa_s = \alpha n_b \theta_{sb} \mathbb{E}\pi + \alpha\left(n_m - n_t\right)\theta_{sm}\Delta_t. \tag{10}$$

[7] For $n \in [0, n_m]$, its path obviously must be nonnegative and bounded. For Δ, boundedness follows from transversality (e.g., see Rocheteau and Wright 2013), while $\Delta \geq 0$ follows because $\Delta < 0$ is inconsistent with free disposal, which is naturally assumed.

[8] The environment is the same with and without M in the sense that it always has endogenous market composition due to S entry. With M entry, we eliminate endogenous composition if we eliminate M, but that case is still covered in Section 4.2. For completeness we tried entry by B, too, but it is less interesting, unsurprisingly, since type B is fairly mechanical here.

This lets us eliminate N_t from (8) and (9), resulting in a two-dimensional system

$$\begin{bmatrix} \dot{\Delta}_t \\ \dot{n}_t \end{bmatrix} = \begin{bmatrix} f(n_t, \Delta_t) \\ g(n_t, \Delta_t) \end{bmatrix}, \tag{11}$$

where f and g are

$$\begin{aligned} f(n_t, \Delta_t) &= -\frac{n_b \kappa_s \theta_{mb}}{n_b \theta_{sb} \mathbb{E}\pi + (n_m - n_t)\,\theta_{sm} \Delta_t} \int_{\Delta_t}^{\infty} [1 - F(\pi)]\, d\pi \\ &\quad + \left[\alpha - \frac{(n_m + n_b)\,\kappa_s}{n_b \theta_{sb} \mathbb{E}\pi + (n_m - n_t)\,\theta_{sm} \Delta_t}\right] \theta_{ms} \Delta_t - \rho + (r + \delta)\,\Delta_t \end{aligned} \tag{12}$$

and

$$\begin{aligned} g(n_t, \Delta_t) &= \left[\alpha - \frac{(n_m + n_b)\,\kappa_s}{n_b \theta_{sb} \mathbb{E}\pi + (n_m - n_t)\,\theta_{sm} \Delta_t}\right] (n_m - n_t) \\ &\quad - \frac{n_b n_t \mathbb{E}\tau(\pi, \Delta_t)\,\kappa_s}{n_b \theta_{sb} \mathbb{E}\pi + (n_m - n_t)\,\theta_{sm} \Delta_t} - \delta n_t. \end{aligned} \tag{13}$$

Equilibrium with entry by S is defined as a nonnegative and bounded path for (Δ_t, n_t) satisfying (11), the initial condition n_0 and the entry condition $rV_{s,t} = \kappa_s$.

Define the n locus and Δ locus as the curves in (n, Δ) space along which $\dot{n} = 0$ and $\dot{\Delta} = 0$, so that they intersect at steady states. We claim both have positive slopes. To verify this, note that while the curves can have kinks (see below), wherever they are differentiable we have:

$$\left.\frac{\partial \Delta}{\partial n}\right|_{\dot{\Delta}=0} = \frac{\theta_{sm} \Delta \left\{ n_b \theta_{mb} \int_{\Delta}^{\infty} [1 - F(\pi)]\, d\pi + (n_m + n_b)\,\theta_{ms} \Delta \right\}}{N \kappa_s D} > 0 \tag{14}$$

$$\left.\frac{\partial \Delta}{\partial n}\right|_{\dot{n}=0} = \frac{n_s + \frac{1}{\kappa_s} \theta_{sm} \Delta \left[\alpha (n_m - n) - \delta n\right] + n_b \left[1 - F(\Delta)\right] + N \frac{\delta}{\alpha}}{\frac{n_m - n}{\kappa_s} \theta_{sm} \left[\alpha (n_m - n) - \delta n\right] + n_b n f(\Delta)} > 0 \tag{15}$$

where

$$\begin{aligned} D &= \left\{ n_b \theta_{mb} \int_{\Delta}^{\infty} [1 - F(\pi)]\, d\pi + (n_m + n_b)\,\theta_{ms} \Delta \right\} (n_m - n)\,\theta_{sm} / N \kappa_s \\ &\quad + N (r + \delta) / \alpha^2 + n_s \theta_{ms} / \alpha + n_b \theta_{mb} \left[1 - F(\Delta)\right] / \alpha. \end{aligned}$$

As both slopes are positive there may be multiple steady states. Although we are actually interested in heterogeneous π, as a preliminary step, consider the degenerate case, $\pi = \bar{\pi}$ with probability 1. In this case there are three possible regimes: (i) M and B trade with probability $\tau = 1$ and $\Delta \leq \bar{\pi}$; (ii) M and B trade with probability $\tau = 0$ and $\Delta \geq \bar{\pi}$; (iii) M and B trade with probability $\tau \in (0, 1)$ and $\Delta = \bar{\pi}$. Consider first $\rho > 0$. Then we have (all proofs are in the Appendix):

Proposition 2 *Suppose $\pi = \bar{\pi}$ with probability 1 and $\rho > 0$. There exists $\tilde{\rho} > 0$ and $\hat{\rho} > \tilde{\rho}$ such that: (i) if $\rho \in [0, \tilde{\rho})$ there is a unique steady state and it has $\Delta < \bar{\pi}$; (ii) if $\rho \in (\hat{\rho}, \infty)$ there is a unique steady state and it has $\Delta > \bar{\pi}$; (iii) if $\rho \in (\tilde{\rho}, \hat{\rho})$ there are three steady states, $\Delta < \bar{\pi}$, $\Delta > \bar{\pi}$, and $\Delta = \bar{\pi}$.*

Example 1: $\alpha = 1$, $\delta = 0.008$, $r = 0.04$, $n_b = 0.05$, $n_m = 0.5$, $\theta_{mb} = 0.7$, $\theta_{sb} = 1$, $\theta_{sm} = 0.5$, $\kappa_s = 0.1$, $\bar{\pi} = 1$, and various ρ.

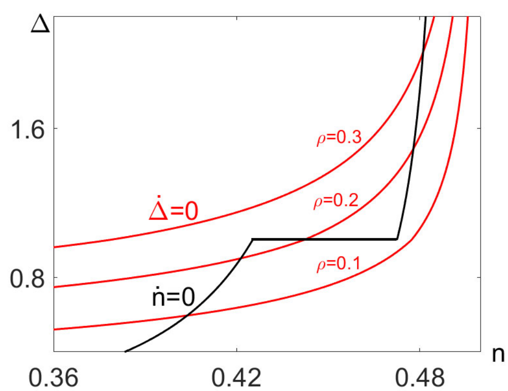

Fig. 1a: Example 1.

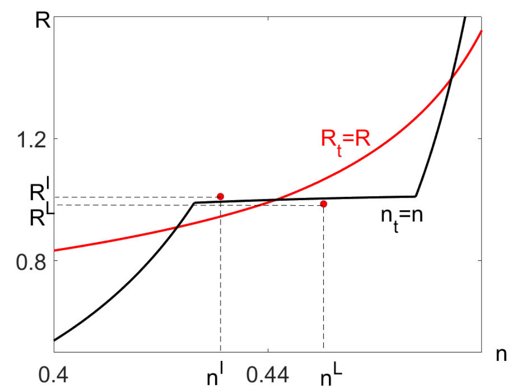

Fig. 1b: Discrete-time version.

Fig. 1 illustrates the result for Example 1 in the left panel (the right panel is for the discrete-time version in Section 4.3; it can be ignored for now). For $\rho = 0.1$ there is one steady state; for $\rho = 0.2$ there are three; and for $\rho = 0.3$ there is one. Multiplicity is explained as above: if Δ is low, M with x trades it to B, so the probability M has x is low, encouraging S entry and making it easy for M to get x, consistent with low Δ; if Δ is high, M with x does not trade x to B, so the probability M has x is high, discouraging S entry, consistent with high Δ. When both $\tau = 1$ and $\tau = 0$ both satisfy the equilibrium conditions, as usual, so does some $\tau \in (0, 1)$.

Notice that market liquidity – i.e., the ease with which agents can buy and sell x – is high (low) when Δ is low (high). Multiplicity means market liquidity is not pinned down by fundamentals, consistent with the idea mentioned earlier that the efficiency/productivity of retail markets differs dramatically across economies

(Lagakos 2016). Also notice that $\tilde{\rho} > 0$ in Proposition 2, meaning steady state is unique for $\rho = 0$. This is also true for $\rho < 0$:[9]

Proposition 3 *Suppose* $\pi = \bar{\pi}$ *with probability* 1 *and* $\rho < 0$. *Then there is a unique steady state and it has* $\Delta < \bar{\pi}$.

This is easy to understand: with π degenerate, M's only alternative to trading x to B is to hang onto x, but such a "buy and hold" strategy makes no sense if $\rho \leq 0$.

These results, with π degenerate, are not at all our main interest. One reason is this: accepting the notion that $\rho > 0$ applies to asset markets and $\rho < 0$ to goods markets, the results mean this kind of multiplicity could emerge in the former but not the latter. We show below that this multiplicity can emerge with $\rho < 0$ when π is not degenerate. Intuitively, nondegenerate π provides an additional motive for M to not trade with B – it can be preferable to wait for a higher π. That does not imply that ρ is irrelevant, of course, since higher ρ makes M and B less inclined to trade for any realization of π, just like, e.g. higher unemployment benefits raise the reservation wage in job-search theory; it does imply that $\rho < 0$ need not preclude multiplicity when π is nondegenerate.

Another reason to move on from π degenerate is that, as shown in Fig. 1a, when there are multiple steady states the middle one lies on the flat segment of the n locus at $\Delta = \bar{\pi}$. This makes it hard to characterize dynamics, while, as shown below, bifurcation methods can be used to good effect once we relax the extreme restriction $\pi = \bar{\pi}$ with probability 1. Yet another reason is this: when π is degenerate, M and B are only indifferent to trade in the rare event $\pi = \Delta$. Hence, if Δ were to vary over time, intermediation activity could vary, too, but with degenerate π we get a bang-bang situation (i.e., τ is almost always 0 or 1). As show below, with disperse π that is still possible, for some parameterizations, but it can also easily be the case that intermediation activity – and hence other endogenous variables – fluctuate smoothly over time.

[9] These results assume M and S trade, which is true if $\rho > 0$, and if $\rho < 0$ but $|\rho|$ is not big, while if $\rho < 0$ and $|\rho|$ is big they do not trade, and any M starting with x disposes of it.

To begin the analysis with disperse π, here is an extension of Proposition 3:

Proposition 4 *Consider a general distribution $F(\pi)$, where $\pi \in [\underline{\pi}, \bar{\pi}]$. If $\rho < 0$ there does not exist a steady state with $\Delta \geq \bar{\pi}$.*

This is also easy to understand: a permanent "buy and hold" strategy still cannot make sense if $\rho \leq 0$, but a "buy and holdout for higher π" strategy can. This is verified by an example:

Example 2: $\alpha = 0.96$, $\delta = 0.001$, $r = 0.01$, $n_b = 0.055$, $n_m = 0.4$, $\theta_{mb} = 0.95$, $\theta_{sb} = 1$, $\theta_{sm} = 0.1$, $\kappa_s = 0.225$, $\rho = -0.014$ and $\pi \sim a\mathcal{N}(b_1, c_1) + (1-a)\mathcal{N}(b_2, c_2)$ with $a = 0.5$, $b_1 = 1$, $b_2 = 2$, $c_1 = 0.01$, and $c_2 = 0.6$.

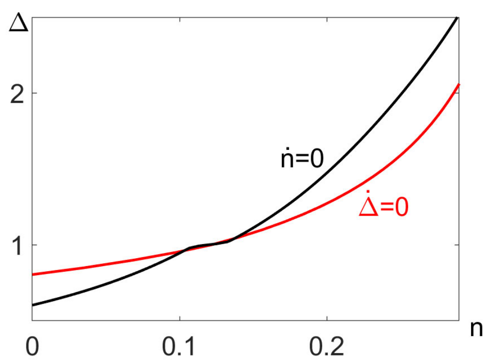

Fig. 2a: Example 2, Steady States.

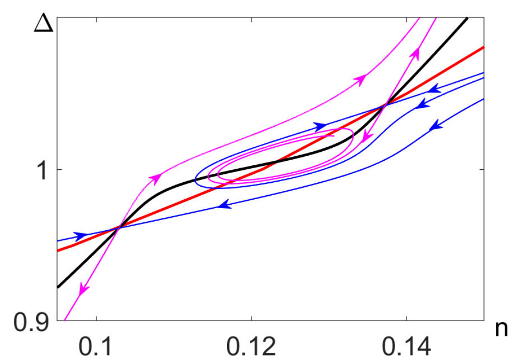

Fig. 2b: Example 2, Phase Plane.

In the left panel, Fig. 2 shows the n and Δ curves for Example 2, where $F(\pi)$ is a mixture of two normal distributions, chosen to illustrate two key points.[10] These points are: the curves can be smooth with no flat segments; and, even with $\rho < 0$, there can be multiple steady states, which here are $(0.103, 0.961)$, $(0.124, 1.005)$ and $(0.137, 1.042)$. In the right panel, Fig. 2 zooms in and shows part of the flow for the dynamic system, although we postpone derivation of that until establishing a few more results about steady state.

[10] In this example the support of π is $(-\infty, +\infty)$, and clearly B and S do not trade in meetings with $\pi < 0$. Hence the above equations need to be modified in an obvious way, left as an exercise, but having support $(-\infty, +\infty)$ actually plays no big role – as usual, we can reinterpret B and S meeting with $\pi < 0$ as B and S not meeting. Examples below use uniform distributions where $\pi > 0$ with probability 1, but we like this one because the curves are globally smooth.

The next result proves existence of steady state for a general $F(\pi)$, and proves the number of steady states is generically odd:[11]

Proposition 5 *If $\rho > -\theta_{mb}\kappa_s/\theta_{sb}$, generically there are an odd number of steady states; they all entail $0 < n < \alpha n_m/(\delta+\alpha)$ and $\Delta > 0$.*

As mentioned for the degenerate case, it is also true here that higher Δ means lower market liquidity. To consider related variables, the average markup μ is the ratio of retail and wholesale prices,

$$\mu = \frac{\frac{\int_\Delta^\infty p_{bm} dF(\pi)}{1-F(\Delta)}}{p_{ms}} = \frac{\int_\Delta^\infty (\theta_{mb}\pi + \theta_{bm}\Delta)\, dF(\pi)}{\theta_{sm}\Delta [1-F(\Delta)]}.$$

The spread σ is the difference between these prices,

$$\sigma = \frac{\int_\Delta^\infty (\theta_{mb}\pi + \theta_{bm}\Delta)\, dF(\pi)}{1-F(\Delta)} - \theta_{sm}\Delta.$$

Trade volume is υ

$$\upsilon = \frac{\alpha n_b n_s}{N} + \frac{\alpha n_b n}{N}[1-F(\Delta)] + \frac{\alpha n_s (n_m - n)}{N}.$$

These are relevant because the markup, spread and volume are used as measures of frictions in both theory (e.g., Weill 2008; Lagos and Rocheteau 2009) and empirical work (e.g., Brennan et al. 1998). Across steady states in Fig. 2a, (μ, σ, υ) are $= (15.491, 1.393, 0.025)$, $(17.656, 1.675, 0.019)$ and $(19.382, 1.915, 0.016)$.[12] Thus at higher Δ, both retail and wholesale prices are higher, but on net the latter effect dominates so that the markup and spread are higher, as is volume, as might be expected in a less liquid market. Later we will investigate how these variables behave over time, not just across steady states.

To begin the dynamics, consider the local properties of steady states. This next result shows that when the Δ curve intersects the n curve from above (below) the steady state is a saddle (spiral):

[11] The condition in Proposition 5 can be understood as follows: If $\rho > 0$, M is willing to hold x both for trading and for its return. If $\rho < 0$ with $|\rho|$ not too big, M is willing to hold x for trading if not for its return, but again, if $\rho < 0$ with $|\rho|$ too big M would not acquire it and would dispose of it if it was in inventory as an initial condition. Now $\rho > -\theta_{mb}\kappa_s/\theta_{sb}$ simply rules out $\rho < 0$ and $|\rho|$ too big.

[12] Because $\pi < 0$ is possible in this example, volume is calculated as $\upsilon = (\alpha n_b n_s/N)[1-F(0)] + (\alpha n_b n/N)[1-F(\Delta)] + \alpha n_s (n_m - n)/N$.

Proposition 6 *If at a steady state* $\partial\Delta/\partial n\Big|_{\dot{\Delta}=0} < \partial\Delta/\partial n\Big|_{\dot{n}=0}$ *the steady state is a saddle point; if* $\partial\Delta/\partial n\Big|_{\dot{\Delta}=0} > \partial\Delta/\partial n\Big|_{\dot{n}=0}$ *the steady state is a spiral that can be locally a sink or a source.*

In the proof of Proposition 5 we show the Δ locus is above the n locus at $n = 0$ and below it at $n = n_m$. Hence, ranking steady states by the value of n, the odd-indexed ones are saddles and the even-indexed ones spirals.

In Fig. 2b, the lower and upper steady states are saddle points and their stable (unstable) manifolds are shown in blue (pink). For these parameters the middle steady state is a sink, with branches of the unstable manifolds of the other steady states spiraling in towards it. Hence, starting from any n_0 in some range, equilibrium can converge to the upper or lower steady state, or can spiral into the middle steady state, depending on initial beliefs about Δ_0.

To go beyond these results, and see what else can happen, consider Fig. 3a, showing the situation for Example 3 below. Again there are three steady states, but now the middle one is a source rather than a sink. Again, starting from any n_0 in some range, there are many equilibria depending on Δ_0, but now we cannot spiral into the middle steady state. This suggests the possibility of cycles, a possibility we now explore using bifurcation theory.[13]

To be clear, the standard method in applications like this establishes the existence of cycles for sets of parameters with positive measure, not merely for numerical examples, if certain conditions hold, and numerical methods are used to verify that these conditions can hold. We also empathize that these conditions generally cannot be verified for all parameters, since cycles can be expected to exist for at most subsets of parameters. Moreover, these conditions generally cannot be verified

[13]References on the dynamical system theory used here include Guckenheimer and Holmes (1983) and Kuznetsov (2004), while Azariadis (1993) is a standard source for economic applications. We employ the Hopf bifurcation, as used to get continuous-time cycles in a search model by, e.g., Diamond and Fudenberg (1989), and the saddle loop bifurcation, used by, e.g., Coles and Wright (1998) or Mortensen (1999). Sniekers (2018) uses the Bogdanov-Takens bifurcation, not previously used in search theory, but used in a macro model by Benhabib et al. (2001). While Sniekers (2018) approach may have some advantages, we find it less tracatbale, and in any case we get what we need with our approach.

in many standard models, since they do not have cyclic equilibria. Proposition 1, e.g., shows that our same environment with one difference – no entry – has a unique equilibrium and no cycles due to self-fulfilling beliefs, as is well known for standard search models like Diamond or Pissarides, as well as Duffie et al. (see Trejos and Wright 2016), without devices like increasing returns or monetary considerations, devices that play no role here.

Example 3 (saddle loop bifurcation): $\pi \sim U\,[0,2]$, $\alpha = 1$, $\delta = 0.001$, $n_b = 0.05$, $n_m = 0.5$, $\theta_{mb} = 0.75$, $\theta_{sb} = 1$, $\theta_{sm} = 0.05$, $\kappa_s = 0.1$, $\rho = 0.108$ and various r.

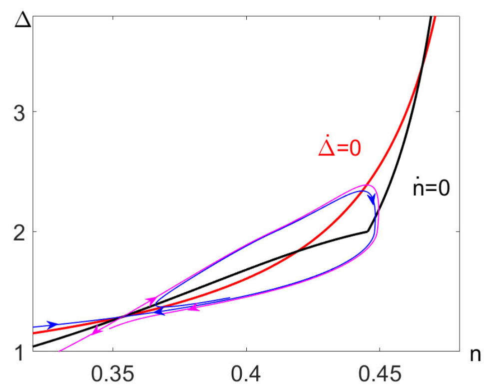

Fig. 3a: Example 3, $r = 0.018$.

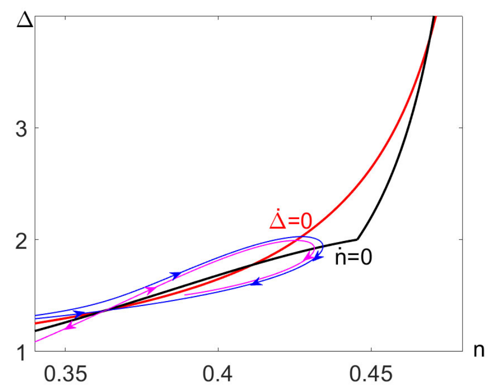

Fig. 3b: Example 3, $r = 0.013$

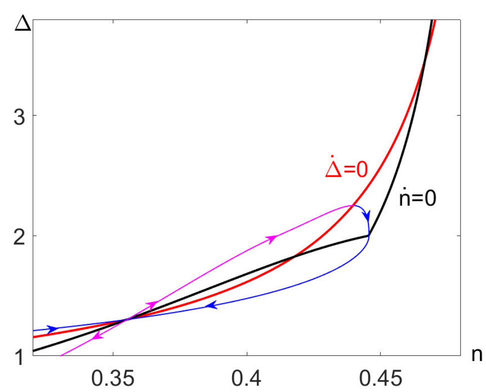

Fig. 3c: Example 3, homoclinic orbit

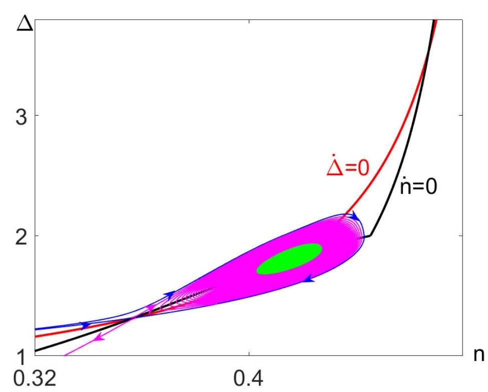

Fig. 3d: Example 3, $r = 0.016$.

The first case involves a saddle loop (also called homoclinic) bifurcation. In Fig. 3a, with $r = 0.018$, the blue stable manifold going to the lower steady state is inside the pink unstable manifold. In Fig. 3b, with $r = 0.013$, the blue stable manifold is outside the pink unstable manifold. By continuity, for some $r^* \in (0.013, 0.018)$ there exists a homoclinic orbit – i.e., the unstable and stable manifold coincide – as shown in Fig. 3c. As the middle steady state inside the homoclinic orbit is a source for these parameters, and any orbit inside the homoclinic orbit cannot escape, the inescapable conclusion is this: starting inside the homoclinic orbit the system must go to a cycle. The green curve in Fig. 3d, for $r = 0.016$, is a trajectory starting near the middle steady state, while the pink curve is the unstable manifold of the lower steady state, and both approach a limit cycle.

The mechanics of saddle loop bifurcations are clear from the graphs, but more formally the Andronov-Leontovich theorem says: Consider a system $\dot{\mathbf{x}} = f(\mathbf{x}, r)$ with $\mathbf{x} \in \mathbb{R}^2$ and a parameter $r \in \mathbb{R}^1$ where f is smooth. Suppose at $r = r^*$ there is a steady state x^* that is a saddle point and has a homoclinic orbit with another steady state inside it. Under mild regularity conditions (Kuznetsov 2004, Section 6.2), $\forall r$ in a nondegenerate neighborhood of r^* there exists a neighborhood of the homoclinic orbit and x^* in which a unique limit cycle bifurcates from the homoclinic orbit (i.e., the cycle emerges as r crosses r^*). The theorem also gives conditions under which cycles are stable or unstable, but the result to emphasize is that they exist for all r in a nondegenerate neighborhood of r^* even if the homoclinic orbit itself exists only at r^*.

Time series from this cycle are shown in Fig. 3e. While the examples are clearly not meant to be calibrations, only to show possibilities, we mention that with $r = 0.016$ a period corresponds to roughly 1 quarter, giving the cycle a not-unrealistic duration of about 7 years. In any case, notice entry and volume lead Δ, while inventories and output lag Δ. Also, the markup (spread) is negatively (positively) correlated with Δ. We also mention that there is a literature on commodity price cycles and how they matter especially in emerging markets.[14] The cycles in these

[14] We thank the referee/editor for suggesting this and providing these reference: Reinhart et al. (2016) document the connection of commodity price and capital flow cycles; Drechsel and Tenreyro (2018) study the impact of commodity prices on the aggregate economy through improvements in competitiveness; Fernandez et al. (2020) argue that world disturbances are responsible for low frequency movements in commodity prices; and Benguria et al. (2024) empirically identify the wealth and cost channels through which commodity cycles affect the economy.

discussions generally result from exogenous shocks. We provide a different account: may be self-fulfilling prophecies.

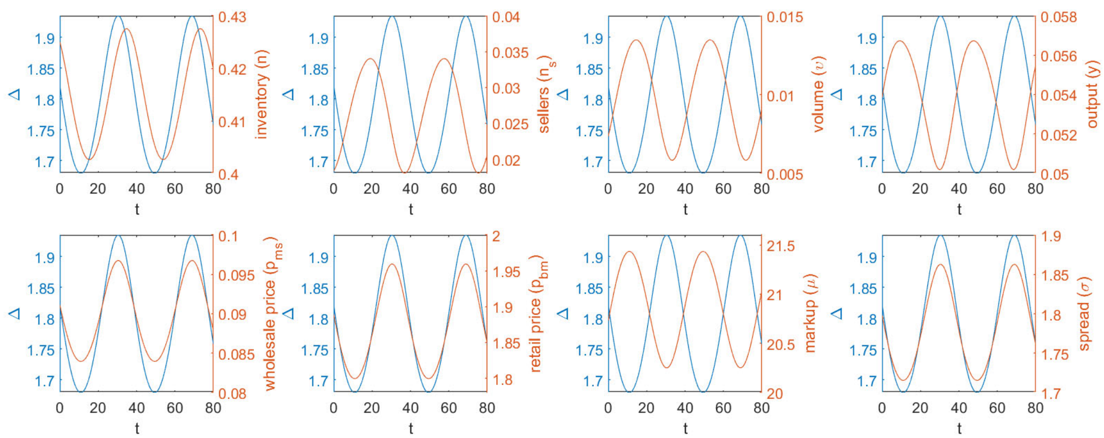

Fig. 3e: Example 3, $r = 0.016$, Time Series.

Example 4 (Hopf bifurcation, subcritical): $\pi \sim U[0,3]$, $\alpha = 1$, $\delta = 0.0001$, $r = 0.0825$, $n_b = n_m = 1$, $\theta_{mb} = \theta_{sb} = \theta_{sm} = 0.5$, $\kappa_s = 0.4$ and $\rho = 0.33$.

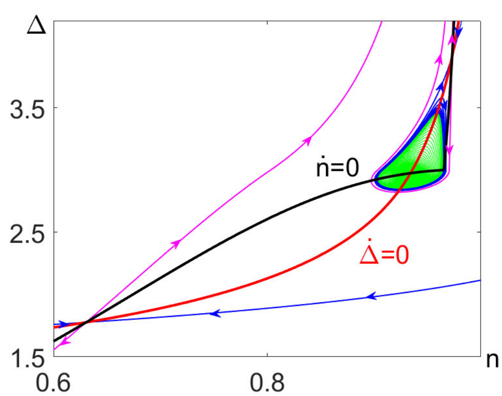

Fig. 4a: Example 4, $r = 0.0825$.

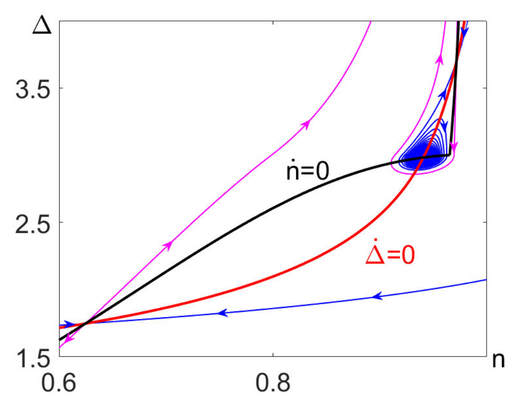

Fig. 4b: Example 4, $r = 0.0875$.

An alternative approach uses the Hopf bifurcation. There are two kinds, supercritical and subcritical, both of which are shown below. Fig. 4 is for Example 4, again with three steady states, and blue (pink) curves showing the stable (unstable) manifolds. The middle steady state can be a sink or a source. As r increases there is a Hopf bifurcation at $r^* = 0.0851$ where the trace of the system is 0: for $r < r^*$ the middle steady state is a sink; for $r > r^*$ it is a source. With $r = 0.0825$ in

Fig. 4a, the stable manifold spirals away from an unstable cycle and goes to the upper steady state, and shown in green is a trajectory spiraling away from the cycle toward the sink. As r increases above r^* the sink becomes a source and the cycle disappears, as shown in Fig. 4b for $r = 0.0875$. In this example the bifurcation is subcritical, meaning a small increase in r around r^* can cause the system to deviate away from the middle steady state.

Fig. 4c plots time series with $r = 0.0825$. Volume, output and entry by S are negatively correlated with Δ, while inventories are positively correlated with Δ. Notice that over the cycle M and B trade with positive probability when $\Delta < 3$, where $\bar{\pi} = 3$ is the upper bound of the support, and do not trade at all when $\Delta > 3$. This can be described as recurrent intermediation freezes and thaws. Actually, the market does not shut down completely during these freezes, since B and S still trade, but M does not trade with B, as they are rationally holding out for better times.[15]

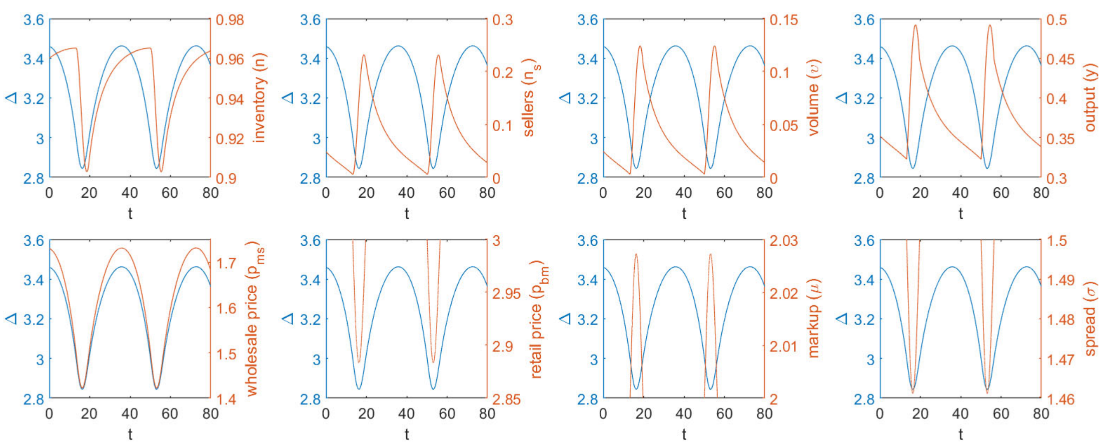

Fig. 4c: Example 4 Time Series.

Example 5 (Hopf bifurcation, supercritical): $\pi \sim U[0, 2]$, $\alpha = 1$, $\delta = 10^{-5}$, $n_b = 2$, $n_m = 1$, $\theta_{mb} = 0$, $\theta_{sb} = 1$, $\theta_{sm} = 0.2$, $\kappa_s = 0.6$ and $\rho = 0.3$.

[15] See Gu et al. (2024) and references therein for discussion of market freezes, with a suggestion that they are interesting and an arguement that it is not easy to get such phenomena in standard models. We also mention that since M does not trade with B during freezes, Fig. 4c only shows the markup and spread during thaws.

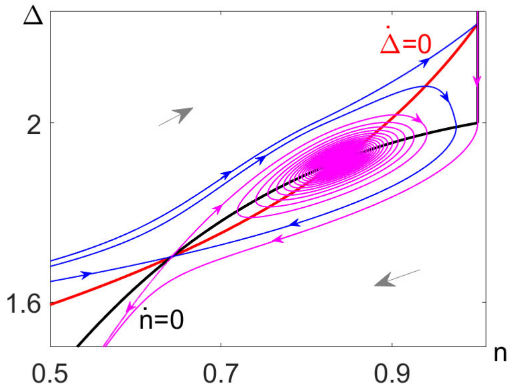

Fig 5a: Example 5, $r = 0.055$.

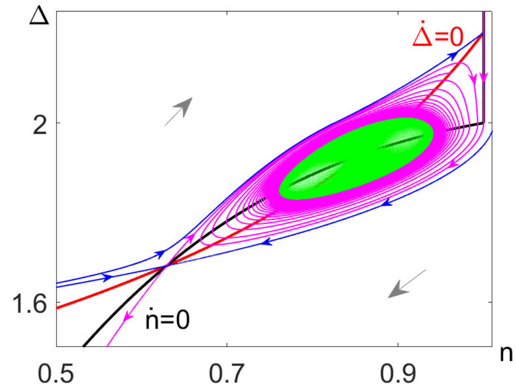

Fig. 5b: Example 5, $r = 0.0562$.

Fig. 5 is for Example 5, with a bifurcation at $r^* = 0.0557$. In Fig. 5a, with $r = 0.055$, the middle steady state is a sink and the unstable manifold of the lower steady state converges to it. As r rises past r^* the sink becomes a source with a stable limit cycle around it. In Fig. 5b, with $r = 0.0562$, the green curve is a trajectory spiraling away from the source, converging to a cycle. The unstable manifolds also converge to a cycle. Fig. 5c plots time series, like Fig. 4c, with a few differences – e.g., the variability of the markup is smaller, and while there are again freezes, they are shorter, and the series are smoother. Also, similar to the saddle loop, with a Hopf bifurcation cycles exist for a set of parameters with positive measure, not just at the bifurcation point r^* (Kutznesov 2004, Theorem 3.4).

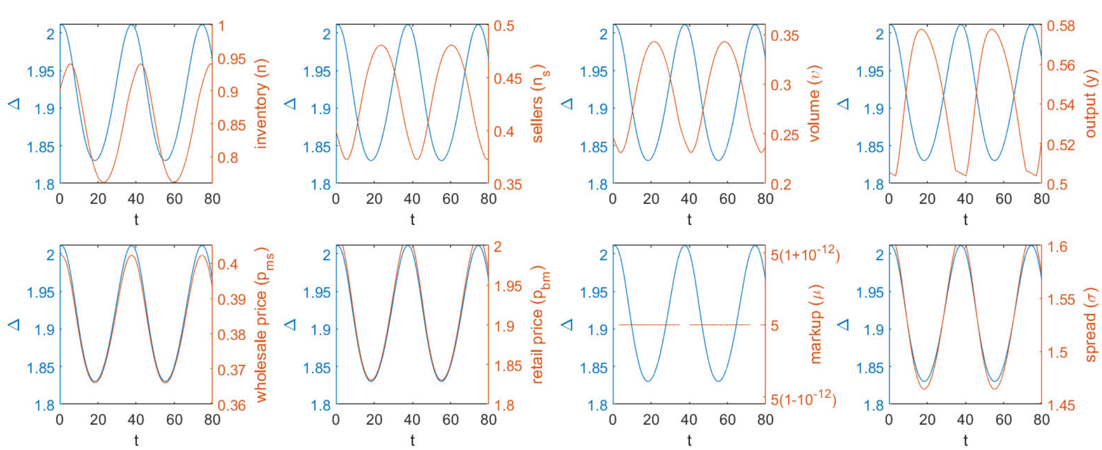

Fig. 5c: Example 5, $r = 0.0562$, Time Series.

Example 6 (another Hopf bifurcation, supercritical): $\alpha = 0.96$, $\delta = 0.001$, $r = 0.01$, $n_b = 0.055$, $n_m = 0.4$, $\theta_{mb} = 0.95$, $\theta_{sb} = 1$, $\theta_{sm} = 0.0843$, $\kappa_s = 0.225$, $\rho = -0.014$, and $F(\pi) = 0$ if $\pi < 0$, and $F(\pi) = 0.5[\Phi(\pi; 1, 0.01) + \Phi(\pi; 2, 0.6) - \Phi(0; 1, 0.01) - \Phi(0; 2, 0.6)]/[1 - 0.5\Phi(0; 1, 0.01) - 0.5\Phi(0; 2, 0.6)]$, where Φ denotes the normal distribution function, if $\pi \geq 0$.

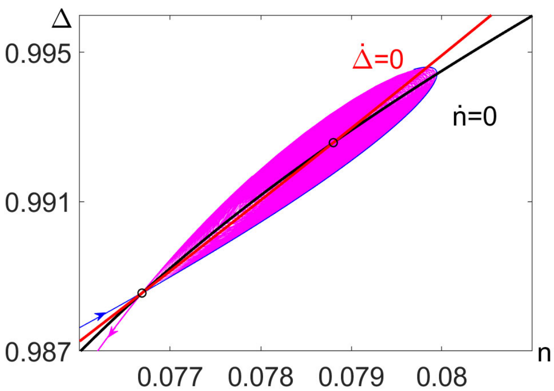

Fig 6a: Example 6, $r = 0.0184$.

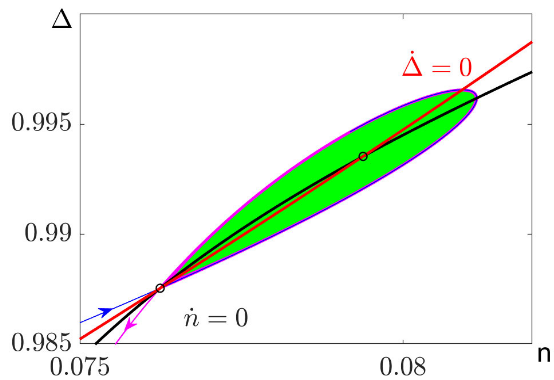

Fig. 6b: Example 6, $r = 0.0185$.

Consider one more case, Example 6, to illustrate another point. The key feature of this case that $\rho < 0$ (the earlier cycles used $\rho > 0$). There are three steady states, but since the interesting dynamics occur near the lower two, which are very close together, Fig. 6, only shows the phase diagram around them. There is a bifurcation at $r^* = 0.01844$. In Fig. 6a, with $r = 0.0184$, the middle steady state is a sink and the unstable manifold of the lower steady state converges to it. As r rises past r^*, as shown in Fig. 6b, the middle steady state becomes a source with a stable limit cycle around it. This cycle is quite small but it still allows us to conclude something important: with disperse π we can not only get multiple steady states, but also limit cycles, with $\rho < 0$.

These findings are summarized as follows:

Proposition 7 *With entry by S, for $\rho > 0$ and for $\rho < 0$, there exists a subset of parameters with positive measure such that for all parameters in the subset there are equilibria with limit cycles.*

In terms of economics, we have shown there are cyclic equilibria due to self-fulfilling expectations about inventories, trading strategies and seller entry, and these exist for subsets of parameters with positive measure. The intuition lies in the strategic complementarity between M and S, as in the case of multiple steady states. When applied to cycles, however, the state variable n and the expectations of the future path of Δ complicate entry decisions. As a result, n, n_s, υ, etc. are not perfectly correlated with Δ, i.e., they may lead or lag Δ.

Our claim is not that actual data are best explained by such cycles in isolation – presumably observations from the real world are driven at least in part by fundamentals, including shocks to technology, policy, etc. We would suggest this: when simple, natural models deliver equilibria with endogenous variables fluctuating due to beliefs, it lends credence to the idea that markets in real economies might be susceptible to similar phenomena. Therefore it is useful to analyze models to see if they display such phenomena. We think that our model of frictional markets with inventories and intermediation is natural, even if highly stylized on some dimensions, and think that the economic ideas are simple, even if some of the mathematics used to get certain results is not.[16]

4 Other Issues

Here we explore some applications and alternative specifications. First we discuss welfare. Then we consider entry by M, instead of S, and show that it leads to uniqueness. Then we ask what changes in discrete time versions of the framework. Then we provide a version of the model with inventories and without middlemen that generates similar results, although it might seem less natural. Finally, we compare in detail this paper with some previous work.

[16] To end this part of the discussion on a tecnical note, we were asked by the referee/editor if increasing dispersion in π shrinks or expands the parameter set where the model admits multiplicity. The answer is that it can go either way. Consider a mean-preserving spread in Example 4, by comparing three cases: (a) $\pi = 2$; (b) $\pi \sim U[1, 3]$; and (c) $\pi \sim U[0, 4]$. For each specification there is a range of ρ giving multiplicity. Moving from case (a) to (b) the range narrows, and from (c) to (d) the range widens.

4.1 Welfare

In the above specification steady state welfare is $W = r\left[n_b V_b + (n_m - n) V_1 + n V_0\right]$, or after inserting the V's

$$W = \frac{\alpha n_s n_b}{N}\theta_{bs}\mathbb{E}\pi + \frac{\alpha n n_b}{N}\int_{\Delta}^{\infty}(\pi - \Delta)\, dF(\pi) + \frac{\alpha n_s (n_m - n)}{N}\theta_{ms}\Delta + n(\rho - \delta\Delta)$$

This includes the surplus when B trades with S, when B trades with M, and when M trades with S, plus the flow payoff from ρ minus the loss from depreciation. Different steady states are distinguished by their Δ. The first term falls with Δ because the number of meetings between B and S falls. The second term is ambiguous because while the surplus in these meetings falls the number of meetings can go either way. The third term is ambiguous for a similar reason. The last term is ambiguous because the total dividend and depreciated value both increase in Δ.

It is easy to construct examples where W decreases with Δ. This is the usual intuition that more liquid markets entail higher welfare. However, in Example 5 above, with $r = 0.0562$, W increases with Δ. Intuitively, while M perform a real service, their activity depends on bargaining power, and they may operate even if it is not socially efficient. Thus, as noted frequently in the literature, W may be higher or lower with middlemen than without them – e.g., Nosal et al. (2015, 2019), Masters (2007,2008), Farboodi et al. (2019) or Gong and Wright (2024). In general, it is well known that ranking welfare across steady states in middlemen models depends on details, including parameter values, so we do not pursue this further here.

Whether welfare is higher or lower in a cycle than in steady state also depends on parameters, plus where we start in the cycle. Suppose there are three steady states. Let W_L, W_M and W_U be welfare in the lower, middle and upper ones, and let W_C be welfare in a cyclic equilibrium. In Example 3, if we start at the highest Δ then $W_L > W_M > W_C > W_U$, and if we start at the lowest Δ then $W_L > W_C > W_M > W_U$. In Example 5, if we start at the highest Δ then $W_U > W_C > W_M > W_L$, while if it starts at the lowest Δ then $W_U > W_M > W_C > W_L$, which is the opposite of Example 3. Intuitively, cycles are bad due to recurrent drops into low-liquidity states, but if we start in a high-liquidity state, that is good, and the net result depends on various factors, including discounting.[17]

4.2 Entry by Middlemen

With endogenous participation by M instead of S, the dynamical system is described by (4)-(9) with constant n_s, time-varying $n_{m,t}$ and entry condition $rV_{0,t} = (n_s/N_t)\,\alpha\theta_{ms}\Delta_t = \kappa_m$. Combining this condition with (8) we get

$$\dot{\Delta} = -\frac{\kappa_m\theta_{mb}n_b}{\theta_{ms}n_s\Delta}\int_{\Delta}^{\infty}\left[1-F\left(\pi\right)\right]d\pi + \kappa_m - \rho + \left(r+\delta\right)\Delta.$$

This is a first-order differential equation, with $d\dot{\Delta}/d\Delta > 0$. Hence, the results are similar to the version with no entry in Section 2: the unique equilibrium has Δ constant at its steady state level. Also,

$$\dot{n} = \alpha n_s - \frac{\kappa_m\left(n_b+n_s+n\right)}{\theta_{ms}\Delta} - \frac{\kappa_m n_b n\left[1-F\left(\Delta\right)\right]}{n_s\theta_{ms}\Delta} - \delta n_t,$$

which implies $d\dot{n}/dn < 0$, so n_t converges to steady state.

While n_t adjusts during the transition Δ_t does not change, the way payoffs do not vary in Pissarides (2000) even while unemployment adjusts to steady state (this is also true in Section 2, but the economics is perhaps more clear here because entry by M is similar to entry by firms in Pissarides-style models). We can also relate the results to Rocheteau and Wright (2005), where buyers choose money balances before entering the market. If seller entry is endogenous, there can be multiple equilibria, since there is a complementarity between buyer and seller strategies; but if buyer rather than seller entry is endogenous, there cannot be multiple equilibria since, heuristically, the same agents make the money holding and entry decisions. A similar intuition applies here, although the complementarity is now between the

[17] Since the editor/referee asked for more on this, let W_0 be welfare with $n_m = 0$, including only the surplus from trade between B and S, and consider a case with three steady states. Letting W_L, W_M and W_U be as in the text, we have: In Example 1, $W_L > W_M > W_U > W_0$. In Example 4 with $r = 0.0825$, $W_L > W_M > W_0 > W_U$. There are also examples with $W_L > W_0 > W_M > W_U$. In Example 5 with $r = 0.0562$, $W_U > W_M > W_L > W_0$, which shows that, when ρ is high, more inventories and a less liquid market can entail higher welfare. The general point is that welfare comparisons here are complicated, but that is not a deficiency in theory – in the real world, intermediated trade in frictional markets is complicated and the models reflect this.

trading decision of M and entry decision of S. In any case we summarize the result as a generalization of Proposition 1:

Proposition 8 *Without entry, or with entry by M, equilibrium is unique: there are no dynamics due to self-fulfilling expectations.*

4.3 Discrete Time

Now consider a discrete-time model, with α the meeting probability, δ the depreciation probability and $\beta \in (0,1)$ the discount factor. The surpluses are

$$\Sigma_{bs,t} = \pi,\ \Sigma_{ms,t} = (1-\delta)\,\beta\Delta_{t+1},\ \Sigma_{bm,t} = \pi - (1-\delta)\,\beta\Delta_{t+1},$$

where again $\Delta_t = V_{1,t} - V_{0,t}$. Now $R_t \equiv (1-\delta)\,\beta\Delta_{t+1}$ is the reservation value satisfying $\Sigma_{mb,t} = 0$. Prices are as in (2) except R_t replaces Δ_t.

Letting $\tau(\pi, R_t)$ be as in (3), the discrete-time value functions are

$$V_{b,t} = \frac{\alpha n_{s,t}}{N_t}\theta_{bs}\mathbb{E}\pi + \frac{\alpha n_t}{N_t}\tau(\pi, R_t)\,\theta_{bm}(\pi - R_t) + \beta V_{b,t+1} \quad (16)$$

$$V_{s,t} = \frac{\alpha n_b}{N_t}\theta_{sb}\mathbb{E}\pi + \frac{\alpha(n_{m,t} - n_t)}{N_t}\theta_{sm}R_t + \beta V_{s,t+1} \quad (17)$$

$$V_{0,t} = \frac{\alpha n_{s,t}}{N_t}\theta_{ms}R_t + \beta V_{0,t+1} \quad (18)$$

$$V_{1,t} = \rho + \frac{\alpha n_b}{N_t}\theta_{mb}\int_{R_t}^{\infty}(\pi - R_t)\,dF(\pi) + (1-\delta)\beta V_{1,t+1} + \delta\beta V_{0,t+1}. \quad (19)$$

Subtracting (18) from (19) and simplifying, we get a difference equation analogous to the differential equation (8),

$$R_{t-1} = (1-\delta)\,\beta\left\{\rho + R_t + \frac{\alpha n_b\theta_{mb}}{N_t}\int_{R_t}^{\infty}[1 - F(\pi)]\,d\pi - \frac{\alpha n_{s,t}\theta_{ms}}{N_t}R_t\right\}. \quad (20)$$

Similarly, we get a law of motion analogous to (9),

$$n_{t+1} = (1-\delta)\,n_t\left[1 - \frac{\alpha n_b}{N_t}\mathbb{E}\tau(\pi, R)\right] + (1-\delta)\,\frac{(n_{m,t} - n_t)\alpha n_{s,t}}{N_t}. \quad (21)$$

With no entry, one can check $dR_t/dR_{t-1} > 1$. Hence (20) has a unique equilibrium, which is the steady state R. Also, $dn_{t+1}/dn_t \in (0,1)$, so n_t converges to the steady state n. Now consider entry by S with a per-period cost, which reduces to exactly (10) in the benchmark model. Given initial n_0, equilibrium is a nonnegative, bounded path for (n_t, R_t) satisfying (20)-(21), written compactly as

$$\begin{bmatrix} R_{t-1} \\ n_{t+1} \end{bmatrix} = \begin{bmatrix} f(n_t, R_t) \\ g(n_t, R_t) \end{bmatrix}.$$

Now the n locus satisfying $n = g(n, R)$ and the R locus satisfying $R = f(n, R)$ both slope up in (n, R) space indicating the possibility of multiplicity.

Example 7: The same as Example 1 plus $\rho = 0.2$.

There are three steady states $(0.9007, 0.4213)$, $(1, 0.4421)$ and $(1.4826, 0.4777)$, similar to the continuous time specification. However, the discrete time dynamics are rather different. Let us focus on a two-cycle, oscillating between a liquid regime with low R and an illiquid regime with high R, denoted (R^L, n^L) and (n^I, R^I). These solve

$$\begin{bmatrix} R^I \\ n^I \end{bmatrix} = \begin{bmatrix} f(n^L, R^L) \\ g(n^L, R^L) \end{bmatrix} \text{ and } \begin{bmatrix} R^L \\ n^L \end{bmatrix} = \begin{bmatrix} f(n^I, R^I) \\ g(n^I, R^I) \end{bmatrix}. \tag{22}$$

A solution is $(R^L, n^L) = (0.9800, 0.4511)$ and $(R^H, n^H) = (1.0065, 0.4297)$, shown in Fig. 1b.

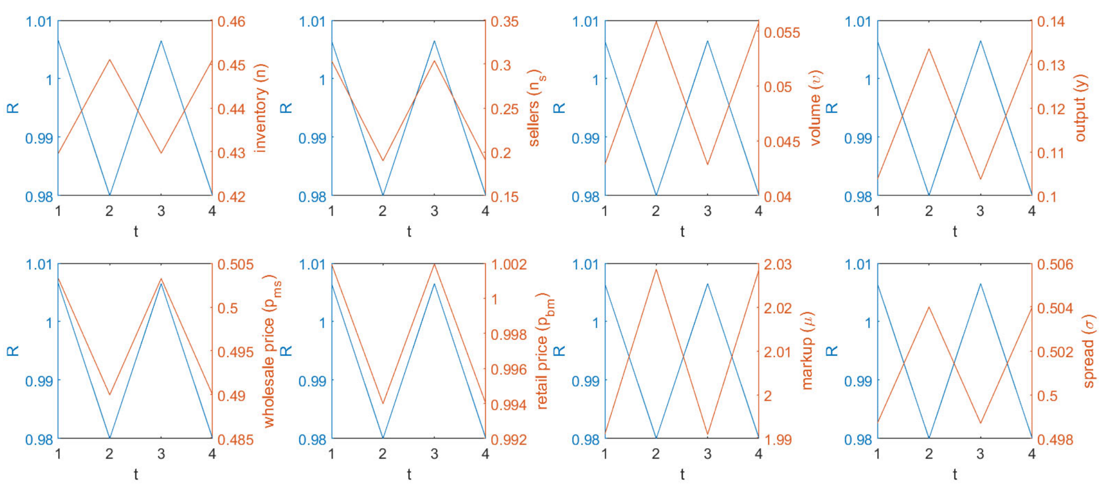

Fig. 7: Example 7, Time Series.

Fig. 7 shows the times series. In the liquid regime R is low, making M more likely to trade with B, and n is high because M and B traded less last period, while

n_s is low because low R and high n discourage entry by S. The illiquid regime is just the opposite.[18]

Next, consider entry by M, which it will be recalled implied uniqueness in continuous time. Now (16)-(21) are the same, but n_s is fixed while $n_{m,t}$ is endogenous. Now (17) yields N_t in terms of R_t,

$$\kappa_m N_t = \alpha n_s \theta_{ms} R_t. \tag{23}$$

From (23), N_t depends only on R_t, while with entry by S it depends on R_t and n_t. Substituting (23) into (20), after some algebra we get $R_{t-1} = G(R_t)$, where

$$G(R) \equiv \beta(1-\delta)\left\{\rho + R + \frac{\kappa_m n_b \theta_{mb}}{n_s \theta_{ms} R}\int_R^\infty [1 - F(\pi)]\, d\pi - \kappa_m\right\}. \tag{24}$$

Now R_{t-1} depends only on R_t, while with entry by S it depends on R_t and n_t.

The univariate system $R_{t-1} = G(R_t)$ determines the path for R_t, from which we get N_t, n_t, etc. Steady state solves $R = G(R)$ as long as it implies $n_m, n \geq 0$, both of which hold iff $R \geq \underline{R} \equiv (n_s + n_b)\kappa_m / \alpha n_s \theta_{ms}$ (we also need $n \leq n_m$ but that never binds). A solution to $R = G(R) \geq \underline{R}$ is a steady state with M active. One can check $G(0) = \infty$, $G'(R) < 1$ and $G''(R) \geq 0$. Also, $\forall R > \bar{\pi}$ G is linear with slope $\beta(1-\delta)$. This is shown in Fig. 8a, from which it is clear that there exists a unique fixed point $\hat{R}$. We can have $\hat{R} > \bar{\pi}$, on the linear part of $G(R)$, or $\hat{R} < \bar{\pi}$, on the nonlinear part. If $G'(\hat{R}) < -1$ then standard methods imply there are cycles. There is a threshold ρ_1 such that $G'(\hat{R}) < -1$ iff $\rho < \rho_1$. We now show that $G'(\hat{R}) < -1$ and $\hat{R} \leq \underline{R}$ are possible.

Example 8: $\pi \sim [0, 0.7]$, $\alpha = 1$, $\delta = 0.01$, $\beta = 0.99$, $n_b = n_s = 1$, $\theta_{mb} = 1$, $\theta_{ms} = 0.1$, $\kappa_m = 0.001$, and various ρ.

Fig. 8a depicts $G(R)$ in Example 8. As ρ decreases, the slope at steady state falls. One can check $G'(\hat{R}) < -1$ and $\hat{R} < \underline{R}$ when $\rho = -0.1$. Hence there is a 2-cycle and possibly cycles of higher order. Fig. 7b plots the second and third iterates,

[18] Prices are also shown in Fig. 6b. The direct price is constant over time, depending only on fundamentals, but the wholesale and retail prices move with R. The spread can go either way, but here it moves against R. This is all broadly consistent with the data discussed in Comerton-Forde et al. (2010), and other stylized facts like inventories being more volatile than output. While this is, again, obviously not a calibration, the finding that it is qualitatively consistent with observations may lend further credence to the story.

$G^2(R)$ and $G^3(R)$. A fixed point of G^2 (G^3) other than a steady state is a 2-cycle (3-cycle). As shown, there exist a pair of 3-cycles. The existence of 3-cycles implies the existence of k-cycles $\forall k$ plus chaotic dynamics, by the well-known Sarkovskii and Li-Yorke theorems

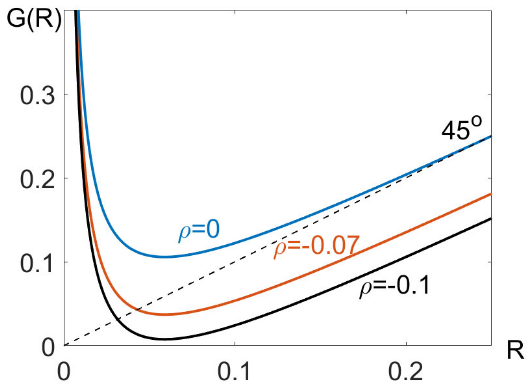

Fig. 8a: Example 8, Different ρ.

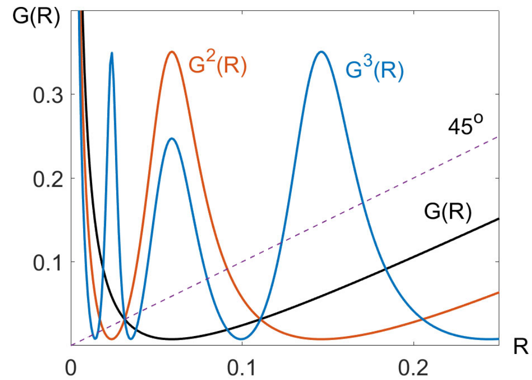

Fig. 8b: Example 8, 2 and 3 Cycles, $\rho = -0.1$.

The discrete-time model is relatively easy to analyze, and generates interesting dynamics with entry by M, counter to the main model. However, these equilibria are not robust: they vanish as the period length shrinks:

Proposition 9 *In the discrete-time model with entry by M, where h denotes the length of a period, there exists $\underline{h} > 0$ such that for all $h \in (0, \underline{h})$ no cycles exist.*

Oberfield and Trachter (2012), Rocheteau and Choi (2021) and Rocheteau and Wang (2023) in different but somewhat related models show that cycles may occur in discrete time but they vanish as the period length shrinks. This is what motivated us to check which results are robust and which are not. While discrete time with entry by M is quite tractable and delivers some interesting results, one might worry this is an artifact of the period length. Discrete time with entry by S is less tractable, but more robust: interesting dynamic equilibria still exist when the period length shrinks, as Section 3 shows – i.e., working directly with continuous time we established the existence of cyclic equilibria.

4.4 Inventories Without Middleman

It turns out that middleman are not strictly necessary for our results; what actually matters is that there are both entry and inventory decisions. The middlemen framework is a natural way to capture this, with entry decisions by S and trading decisions by M. Still, an environment can be rigged with no type M, so S and B must trade directly, but now B has the option to consume x for payoff π or store it for return ρ, which plays the role of M's option to trade x or store it in the main model. One can interpret storage by B as savings, as opposed to consumption. This inventory/savings option is only viable when $\rho > 0$, which is a reason one might prefer the specification with M. In any case, the purpose of this extension is to show it is possible to get similar results without M, even if the model with M seems better on some dimensions.

As usual B's payoff is match specific, $\pi \sim F(\pi)$, and B and S trade as long as $\pi > 0$. If B chooses not to consume x, it is inventoried for flow return ρ and depreciates at rate δ. Assume for simplicity that if B decides to store x the decision is irreversible – it is not possible to later consume it and go back on the market, so B is off the market until x depreciates. This restriction does not bind in, but could bind out of, steady state. While it is not especially natural, it lets us make the point relatively easily. Note that no such restriction is needed in the model with M, which may be another reason to prefer that version.

Normalize the measure of B to 1, and let S enter the market by paying κ_s. Let V_s, V_0 and V_1 be the value functions of S, B without inventory and B with inventory. When trading with S, if B inventories x the surplus is $\Delta = V_1 - V_0$, and if B consumes x the surplus is π. Then B consumes x if π exceeds the reservation value $R = \Delta$. Let θ be B's bargaining power. Then

$$\begin{aligned} rV_s &= \frac{\alpha(1-n)}{1+n_s}(1-\theta)\left[\int_0^R \Delta dF(\pi) + \int_R^{\bar{u}} \pi dF(\pi)\right] + \dot{V}_s \\ rV_0 &= \frac{\alpha n_s}{1+n_s}\theta\left[\int_0^R \Delta dF(\pi) + \int_R^{\bar{u}} \pi dF(\pi)\right] + \dot{V}_0 \\ rV_1 &= \rho - \delta\Delta + \dot{V}_1. \end{aligned}$$

This leads to

$$\dot{\Delta} = (r+\delta)\Delta - \rho + \frac{\alpha n_s}{1+n_s}\theta\left\{\Delta + \int_{\Delta}^{\bar{u}} [1-F(\pi)]\, d\pi\right\},$$

after integration by parts. The law of motion for inventories is

$$\dot{n} = -\delta n + \frac{\alpha(1-n)n_s}{1+n_s}F(\Delta)$$

and the entry condition by S implies

$$\frac{\alpha(1-n)}{1+n_s}(1-\theta)\left\{\Delta + \int_{\Delta}^{\bar{u}} [1-F(\pi)]\, d\pi\right\} = \kappa_s.$$

Consider first a degenerate distribution of π. That leads to

$$\begin{aligned}
\dot{\Delta} &= (r+\delta)\Delta - \rho + \frac{\alpha n_s}{1+n_s}\theta\left[\gamma\pi + (1-\gamma)\Delta\right] \\
\dot{n} &= -\delta n + \frac{\alpha(1-n)n_s}{1+n_s}(1-\gamma) \\
\kappa_s &= \frac{\alpha(1-n)}{1+n_s}(1-\theta)\left[\gamma\pi + (1-\gamma)\Delta\right]
\end{aligned}$$

where γ denotes B's probability of consuming x. In terms of steady state, there are three regimes where $n_s > 0$ (i.e., where the market does not shut down), $\gamma = 1$, $\gamma = 0$ and $\gamma \in (0,1)$, plus a regime with $n_s = 0$. In the Appendix we construct the set of parameters that make γ a best response to itself, and check whether $n_s > 0$, $n_s = 0$ or $n \in [0,1]$.

Example 9: $\alpha = 0.1$, $\delta = 0.01$, $n_b = 1$, $\theta = 0.73$, $\kappa_s = 0.01$, various π and ρ.

The results are shown in Fig. 9a, partitioning parameter space into 4 regions that support the different regimes. The pattern is general, and its properties are derived as Claim 1 in the Appendix, while the picture is drawn for the specification in Example 9. In the gray area $n_s = 0$, so the market shuts down. Otherwise, $n_s > 0$ and: in the blue region $\gamma = 1$ since π is high relative to ρ; in the brown region $\gamma = 0$ since π is low relative to ρ; in the green region there are three steady states, $\gamma = 0$, $\gamma = 1$ and $\gamma \in (0,1)$. Hence multiple steady states can exist, as in the main model, with M, with a similar intuition: if γ is high B is often without

x, making it easier for S to trade, making n_s big and B more inclined to consume x; but if γ is low, and so on.

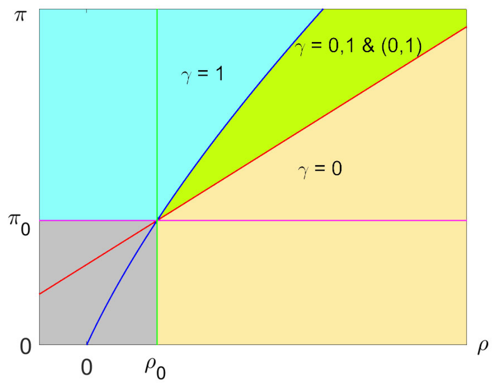

Fig. 9a: Regions/regimes without M.

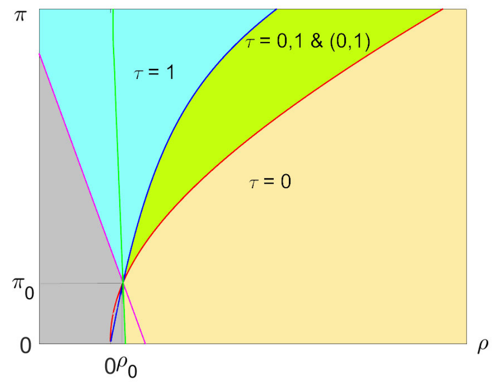

Fig. 9b: Regions/regimes with M.

If Fig. 9a is useful, Fig. 9b provides a similar picture for the model with M, drawn for Example 1 in Section 3. However, the interpretation is: here S and B always trade; S and M do not trade in the grey area and trade in other regions; and those other regions differ in the probability τ that M trades with B. We did not draw this graph earlier because deriving the regions with three types is more cumbersome – Fig. 9b is done numerically – and because these pictures are only relevant for degenerate π, while the preferred specification has disperse π. Still, the figures are remarkably similar, and Fig. 9b nicely tightens a loose end in Section 3, where it was simply assumed that ρ is above some lower bound to guarantee M and S trade; now the boundary of grey area tells us just how low ρ can go before S and stop M trading. More generally, these pictures indicate that steady state exists for all parameters, although for some parameters certain agents stop trading.

Example 10: $a = 0.3$, $b = 0.31$, $c = 0.48$, $d = 0.5$, $y_1 = 0.01$, $y_2 = 0.49$, $y_3 = 0.5$, $\alpha = 0.46$, $\delta = 0.046$, $n_b = 1$, $\theta = 0.75$, $\kappa_s = 0.01$ and $\rho = 0.1$

Going beyond steady state, the model without M also has cyclic equilibria. As r varies in Example 10 there is a Hopf bifurcation at $r^* = 0.0174$. Fig. 10a shows $r = 0.015$, where there are four steady states with the lowest being a sink, and the unstable manifold of the next lowest steady state converges to the lowest one. As

r increases past r^* the sink becomes a source, with a stable limit cycle around it. Fig. 10b shows the case with $r = 0.020$, with the green curve showing a trajectory spiraling away from the source and converging to a cycle. The unstable manifold also converges to a cycle. Since cycles emerge when r increases, $r^*$is supercritical.

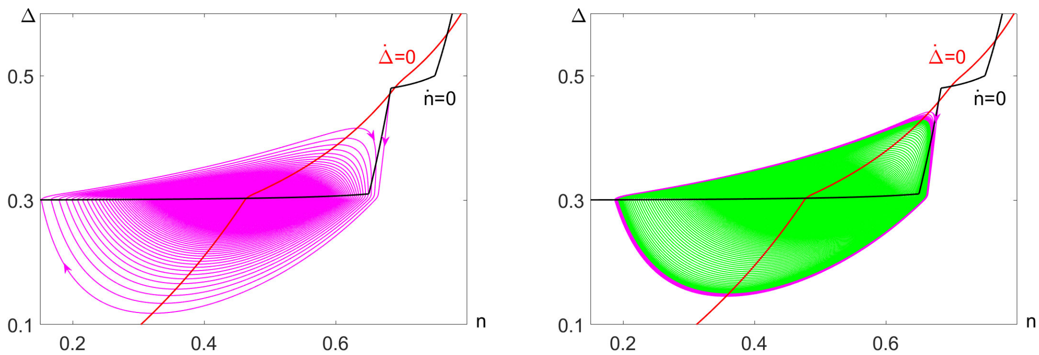

Fig. 10a: Example 10, $r = 0.015$. Fig. 10b: Example 10, $r = 0.020$.

We can again check the general intuition, offered above, that multiplicity and cycles emerge when agents on one side make an inventory decision while those on the other side make an entry decision. The Appendix considers a version without M, where B makes both entry and inventory decisions, and proves as Claim 2 that equilibrium is unique. It also considers a version without M, where S makes both entry and inventory decisions and proves as Claim 3 that equilibrium is unique. These results are consistent with intuition.

To summarize, clearly we cannot claim that middlemen are necessary for our results – we just showed that what one really needs are inventories combined with certain entry decisions. Our claim is that models of intermediated trade are a natural way to think about inventories, and that provide an interesing connection from middlemen to multiplicity and endogenous dynamics.

4.5 A Comparison

Here we compare our paper to one that is close in some respects, Nosal et al. (2019), hereafter NWW.[19] While NWW also have sellers S, middlemen M, and buyers B,

[19]This discussion was suggested by the editor/referee as a way to highlight our contribution.

there are many differences in the details that matter for the economics and for the mathematics.[20] For present purposes, the most important difference is this: in NWW π is constant, while we have a distribution of valuations $F(\pi)$. With degenerate π, in dynamic equilibria, at any point in time either M and B for sure trade or for sure do not trade except for the knife-edge case $\Delta_t = \pi$ – this is what is mean by a bang-bang solution. That leads to problems:

- Multiplicity is possible only for $\rho > 0$.
- If there are multiple steady states, there must be one above, one below, and one on the horizontal line as Fig. 1a, reproduced here for convenience, in a stylized way, as Fig. 11a. For comparison, in our model, when there are three steady states, the situation looks like Fig. 11b.

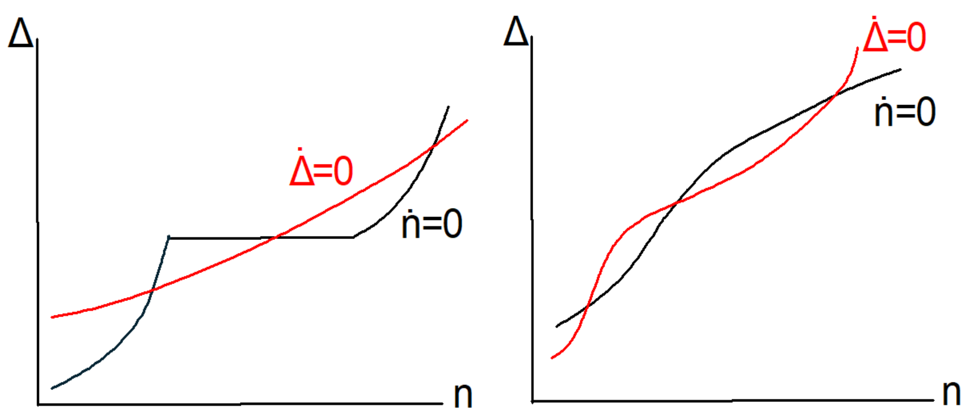

Fig. 11a: NWW Fig. 11b: GWW

The point to emphasize is that in our model the question is not *if* M sells to B, but *when* M sells to B, depending on the realized match-specific valuation π. As shown above, the optimal stopping rule is a reservation strategy: B and M trade when $\pi \geq R$, where R is a *smooth* function of exogenous and endogenous variables. This implies:

[20]There too many differences to list them all, but in addition to what is emphasized in the text, we mention that in our model, as is standard in search theory, a key ingredient is entry by S. NWW do not have entry but let agents choose their occupation, M or S, which is problematic for several reasons. One is that in their setup, in dynamic equilibrium, some agents are continuously changing occupations; here agents never change occupations (types) but participate rates fluctuate. Another is that in their setup $B + S + M$ is fixed, while we find it interesting to have the size of the market endogenous.

- Multiple steady states and cycles are possible for $\rho > 0$ and for $\rho < 0$.
- We can apply bifurcation theory to our dynamical system.
- In particular, NWW do not prove anything about the existence of cycles. As they say "Whether [an example they consider] converges to steady state, or to a small cycle around it, is hard to say from the numerical output, and checking local stability directly is hindered by the [system] being nondifferentiable." In contrast, we prove the existence of cyclic equilibria for sets of positive measure in parameter space.

5 Conclusion

This paper studied dynamic models of inventories, focusing on intermediated trade, heterogeneous buyer valuations, and endogenous entry. We showed there are multiple steady states and endogenous cycles, where entry, liquidity, prices and other endogenous variables fluctuate. This is driven by strategic considerations, not increasing returns, or the self-referential nature of acceptability, as in some other models. We analyzed discrete- and continuous-time specifications. In some cases (entry by middlemen) discrete time was tractable and gave interesting results, but they are not robust to period length; in other cases (entry by sellers), cyclic equilibria are possible in discrete and continuous time.

As mentioned above, the relevance of the findings is this: while it may be hard to account for data based purely on self-fulfilling prophecies, when simple and natural models display such outcomes, it may make one more inclined to think that actual economies can, too. This is consistent with Diamond's (1982b) view, but to get the desired results he needed increasing returns, which he called an externality. As he said, "this externality involves positive feedback: increased production for inventory makes trade easier; easier trade makes production for inventory more profitable and therefore justifies its increase. This positive feedback... implies the possibility of multiple equilibria." His models do not capture inventory behavior the way we do

– there are no middlemen – but, interpreted broadly, the spirit is similar.

One might say that when strategic considerations that arise naturally with intermediation are introduced, we can dispense with increasing returns and still get multiplicity and complicated dynamics. To put this in context, Diamond (1984) got money into his framework with a CIA (cash-in-advance) restriction, and again increasing returns led to multiplicity. Subsequent developments showed that modeling the microfoundations in more detail means the CIA constraint is not needed to get valued fiat currency, and that further implies increasing returns are not needed to get natural multiplicities and dynamics in monetary economics, as emphasized by, e.g., Kiyotaki and Wright (1993) or Johri (1999). What we think is a general conclusion is that markets with frictions are prone to volatility or instability, in the sense that there can be multiple steady states and cyclic dynamic equilibria arising as self-fulfilling prophecies. This is well know in monetary models. Our results show that real models with inventories have similar properties.

In other words, once the exchange process is modeled in more detail, and exchange is not as simple as it is in Diamond's early work, where everyone always wants to trade with everyone else, there is a role for institutions that facilitate this process. Two such institutions are money and middlemen. Once they are modeled explicitly, one can dispense with mechanical assumptions like increasing returns and still get interesting results. We find it interesting that money and middlemen are similar in that regard.

Appendix

Proof of Proposition 2: First notice that, with $\pi = \bar{\pi}$, (8) and (9) reduce to

$$(r+\delta+\alpha\theta_{ms})\Delta - \rho - \frac{\alpha n_b \theta_{mb} \tau (\bar{\pi}-\Delta) + \alpha (n_b+n_m) \theta_{ms} \Delta}{N} = 0 \quad (25)$$

$$\delta n + \frac{\alpha n n_b \tau}{N} - \alpha(n_m - n)\left(1 - \frac{n_b+n_m}{N}\right) = 0 \quad (26)$$

where $N = [\alpha n_b \theta_{sb} \bar{\pi} + \alpha (n_m - n) \theta_{sm} \Delta] / \kappa_s$. In the region where $\Delta > \bar{\pi}$, with $\tau = 0$, we combine (25) and (26) to eliminate N,

$$\left(r+\delta+\frac{\theta_{ms}\delta n}{n_m - n}\right)\Delta = \rho. \quad (27)$$

This implies

$$\frac{\partial \Delta}{\partial n} = -\frac{\theta_{ms}\delta n_m \Delta}{(n_m-n)^2 (r+\delta) + (n_m - n)\theta_{ms}\delta n} < 0.$$

This transforms (25)-(26) to an equivalent system (26)-(27). As (27) is downward sloping and (26) upward sloping, there is at most one steady state with $\Delta > \bar{\pi}$. Also, from (27), steady state exists in this region only if $\rho > 0$.

In the region where $\Delta < \bar{\pi}$, with $\tau = 1$, we combine (25) and (26) to get

$$(r+\delta+\alpha\theta_{ms})\Delta = \rho + \frac{n_b \theta_{mb}(\bar{\pi}-\Delta) + (n_b+n_m)\theta_{ms}\Delta}{n_m(n_b+n_m-n)}\left[\alpha(n_m-n) - \delta n\right].$$

This implies

$$\frac{\partial \Delta}{\partial n} = -\frac{n_b \theta_{mb}(\bar{\pi}-\Delta) + (n_b+n_m)\theta_{ms}\Delta}{r+\delta+\frac{\theta_{ms} n[\alpha n_b + \delta(n_m+n_b)] + \theta_{mb} n_b [\alpha(n_m-n)-\delta n]}{n_m(n_b+n_m-n)}} \frac{\delta(n_b+n_m)+\alpha n_b}{n_m(n_b+n_m-n)^2} < 0. \quad (28)$$

Again, since (28) is downward and (26) upward sloping, there is at most one steady state with $\Delta < \pi$. Similarly, when $\Delta = \bar{\pi}$ and $\tau \in (0,1)$, the n locus is flat and Δ locus upward sloping. Hence, there again is at most one steady state.

For existence, it is easily verified that Δ and n loci are upward sloping and the Δ locus is flatter than the n locus in regions where $\Delta \neq \bar{\pi}$. Also, the Δ locus shifts up when ρ increases. For $\rho = -\theta_{mb}\kappa_s/\theta_{sb}$, the Δ locus goes through the origin. For $\rho > -\theta_{mb}\kappa_s/\theta_{sb}$, the Δ locus has a positive intercept. At $n = n_m$, Δ is positive and finite on the Δ locus. The n locus goes through $(\underline{n}, 0)$, where

$$\underline{n} \equiv \alpha n_m \left(1 - \frac{n_b+n_m}{\alpha n_b \theta_{sb}\bar{\pi}}\kappa\right) \Big/ \left[\delta + \frac{\kappa}{\theta_{sb}\bar{\pi}} + \alpha\left(1 - \frac{n_b+n_m}{\alpha n_b \theta_{sb}\bar{\pi}}\kappa\right)\right],$$

strictly first increases, then becomes flat and goes to ∞ as n goes to $\alpha n_m / (\alpha + \delta) < n_m$. Hence the loci have at least one intersection. In particular, if there is a steady state at $\Delta = \bar{\pi}$, there are two more steady states, one with $\Delta < \bar{\pi}$ and one with $\Delta > \bar{\pi}$. As ρ shifts the Δ locus, there exist $\tilde{\rho}, \hat{\rho} \geq 0$ with the stated properties. ■

Proof of Proposition 3. Suppose M buys x from S and does not sell it to B. If $\rho < 0$ this strategy has a negative payoff, which is dominated by not buying x. ■

Proof of Proposition 4. Suppose instead there is a steady state where $\Delta \geq \bar{\pi}$. By (8), $\dot{\Delta} = 0$ implies $(\alpha n_s \theta_{ms}/N + r + \delta)\Delta = \rho$. The LHS is positive while the RHS is negative. A contradiction. ■

Proof of Proposition 5. We prove by the intermediate value theorem. Let $\Delta(n)|_{\dot{\Delta}=0}$ and $\Delta(n)|_{\dot{n}=0}$ denote the Δ and n loci, respectively.

First, substitute $n = 0$ into $f(0, \Delta) = 0$ to solve $\Delta(0)|_{\dot{\Delta}=0}$:

$$\begin{aligned} 0 \quad = \quad & -\frac{\theta_{mb}\kappa_s}{\theta_{sb}\mathbb{E}\pi + n_m\theta_{sm}\Delta}\int_{\Delta}^{\infty}[1 - F(\pi)]\,d\pi \\ & -\frac{[(n_m + n_b)\kappa_s - \alpha(n_b\theta_{sb}\mathbb{E}\pi + \Delta n_m\theta_{sm})]\theta_{ms}}{\frac{n_b\theta_{sb}\mathbb{E}\pi}{\Delta} + n_m\theta_{sm}} - \rho + (r + \delta)\Delta \end{aligned} \tag{29}$$

By (10), for n_s to be positive in equilibrium, we have

$$(n_m + n_b)\kappa_s < \alpha(n_b\theta_{sb}\mathbb{E}\pi + n_m\theta_{sm}\Delta) \tag{30}$$

So the RHS of (29) is increasing in Δ. The RHS at $\Delta = 0$ equals $-\theta_{mb}\kappa_s/\theta_{sb} - \rho$. If $\rho > -\theta_{mb}\kappa_s/\theta_{sb}$, the solution to (29) must entail $\Delta > 0$. That is, $\Delta(0)|_{\dot{\Delta}=0} > 0$.

Next, we show $\Delta(n)|_{\dot{\Delta}=0}$ is bounded. Note that $f(n, \infty) = \infty \neq 0$. Therefore, $\Delta = \infty$ cannot be part of the Δ locus. It follows that $\Delta(n)|_{\dot{\Delta}=0} < \infty$ for any $n \in [0, n_m]$.

Next, check $\Delta(n)|_{\dot{n}=0}$. Substitute $n = 0$ into $g(n, \Delta) = 0$ to solve $\Delta(0)|_{\dot{n}=0}$:

$$0 = \alpha - \frac{(n_m + n_b)\kappa_s}{n_b\theta_{sb}\mathbb{E}\pi + n_m\theta_{sm}\Delta}$$

By (30), $\Delta(0)|_{\dot{n}=0} < 0$. This implies that the n locus has a positive intercept on the n axis. Substitute $\Delta = \infty$ into $g(n, \Delta) = 0$ to get

$$0 = \alpha(n_m - n) - \delta n$$

which results in $n = \hat{n} \equiv \alpha n_m / (\delta + \alpha) < n_m$. That is, $\Delta(\hat{n}) = \infty$.

As $\Delta(n)|_{\dot{\Delta}=0}$ and $\Delta(n)|_{\dot{n}=0}$ are continuous, $\Delta(0)|_{\dot{\Delta}=0} > \Delta(0)|_{\dot{n}=0}$ and $\Delta(\hat{n})|_{\dot{\Delta}=0} < \Delta(\hat{n})|_{\dot{n}=0}$, by the intermediate value theorem, there are generically an odd number of intersections in $n \in (0, \hat{n})$.

Furthermore, there does not exist any steady state with $\Delta \leq 0$ if $\rho > -\theta_{mb}\kappa_s/\theta_{sb}$. It is obvious that if $V_1 < V_0$, the middlemen should dispose of x, so Δ cannot be negative. Suppose $\Delta = 0$. Then $V_0 = 0$ by (6). If a middleman holds x, $rV_1 = \theta_{mb}\kappa_s/\theta_{sb} + \rho > 0$, which contradicts $\Delta = 0$. Therefore, there does not exist any steady state with $\Delta \leq 0$. ■

Proof of Proposition 6. The Jacobian matrix is:

$$J = \begin{bmatrix} \partial\dot{\Delta}/\partial\Delta & \partial\dot{\Delta}/\partial n \\ \partial\dot{n}/\partial\Delta & \partial\dot{n}/\partial n \end{bmatrix}$$

where

$$\begin{aligned}
\frac{\partial\dot{\Delta}}{\partial\Delta} &= \frac{\alpha^2}{N^2\kappa}\theta_{sm}(n_m - n)\left[n_b\theta_{mb}\int_{\Delta}^{\infty}[1-F(\pi)]\,d\pi + (n_m+n_b)\theta_{ms}\Delta\right] \\
&\quad + \frac{\alpha n_b}{N}\theta_{mb}[1-F(\Delta)] + \frac{\alpha n_s}{N}\theta_{ms} + (r+\delta) > 0 \\
\frac{\partial\dot{\Delta}}{\partial n} &= -\frac{\alpha^2}{N^2\kappa}\theta_{sm}\Delta\left[n_b\theta_{mb}\int_{\Delta}^{\infty}[1-F(\pi)]\,d\pi + (n_m+n_b)\theta_{ms}\Delta\right] < 0 \\
\frac{\partial\dot{n}}{\partial\Delta} &= \frac{\alpha^2}{N^2\kappa}\left\{(n_m+n_b)(n_m-n) + n_b n[1-F(\Delta)]\right\}\theta_{sm}(n_m-n) + \frac{\alpha}{N}n_b n f(\Delta) > 0 \\
\frac{\partial\dot{n}}{\partial n} &= -\frac{\alpha n_s}{N} - \delta - \frac{\alpha^2}{N^2\kappa}\left\{(n_m+n_b)(n_m-n) + n_b n[1-F(\Delta)]\right\}\theta_{sm}\Delta < 0
\end{aligned}$$

Totally differentiating, we obtain $\left.\frac{\partial\Delta}{\partial n}\right|_{\dot{\Delta}=0} = \frac{-\partial\dot{\Delta}/\partial n}{\partial\dot{\Delta}/\partial\Delta}$ and $\left.\frac{\partial\Delta}{\partial n}\right|_{\dot{n}=0} = \frac{-\partial\dot{n}/\partial n}{\partial\dot{n}/\partial\Delta}$. If $\left.\frac{\partial\Delta}{\partial n}\right|_{\dot{\Delta}=0} < \left.\frac{\partial\Delta}{\partial n}\right|_{\dot{n}=0}$ then

$$\frac{-\partial\dot{\Delta}/\partial n}{\partial\dot{\Delta}/\partial\Delta} < \frac{-\partial\dot{n}/\partial n}{\partial\dot{n}/\partial\Delta} \Rightarrow -\frac{\partial\dot{\Delta}}{\partial n}\frac{\partial\dot{n}}{\partial\Delta} < -\frac{\partial\dot{\Delta}}{\partial\Delta}\frac{\partial\dot{n}}{\partial n},$$

which means

$$det(J)|_{(\bar{\Delta},\bar{n})} = \frac{\partial\dot{\Delta}}{\partial\Delta}\frac{\partial\dot{n}}{\partial n} - \frac{\partial\dot{\Delta}}{\partial n}\frac{\partial\dot{n}}{\partial\Delta} < 0.$$

This implies that $\lambda_1\lambda_2 < 0$, where λ_1 and λ_2 are the two eigenvalues of J at this steady state, meaning that there must be one positive and one negative. Therefore the steady state is a saddle point.

Similarly, if $\partial\Delta/\partial n\Big|_{\dot{\Delta}=0} < \partial\Delta/\partial n\Big|_{\dot{n}=0}$ then $\lambda_1\lambda_2 > 0$, meaning the two eigenvalues are either both positive or are complex conjugates. Combined with the phase diagram, this establishes that the steady state is a spiral. The stability of the spiral is determined by

$$tr(J)|_{(\bar{\Delta},\bar{n})} = \frac{\partial\dot{\Delta}}{\partial\Delta} + \frac{\partial\dot{n}}{\partial n} = \lambda_1 + \lambda_2.$$

Since $\partial\dot{\Delta}/\partial\Delta > 0$ and $\partial\dot{n}/\partial n < 0$, it can be a source or a sink, as our examples show. ■

Proof of Proposition 9. Let the length of a period be h. Then β, α, κ_m, ρ and δ are functions of h. As usual, let:

$$r = \lim_{h\to 0}\frac{\beta(h)^{-1}-1}{h},\ \alpha = \lim_{h\to 0}\frac{\alpha(h)}{h},\ \kappa_m = \lim_{h\to 0}\frac{\kappa_m(h)}{h},\ \rho = \lim_{h\to 0}\frac{\rho(h)}{h},\ \delta = \lim_{h\to 0}\frac{\delta(h)}{h}$$

The equilibrium condition can be rewritten $R_{t-h} = G(R_t, h)$, where

$$G(R;h) \equiv \beta(h)\left[1-\delta(h)\right]\left\{\rho(h) + R + \frac{\kappa_m(h)\, n_b\theta_{mb}}{n_s\theta_{ms}R}\int_R^\infty \left[1-F(\pi)\right]d\pi - \kappa_m(h)\right\}. \tag{31}$$

As $h \to 0$, this converges to the continuous-time model.

First we show steady state is unique. From (31) we get

$$\frac{\partial G}{\partial R} = \beta(h)\left[1-\delta(h)\right]\left\{1 + \frac{\kappa_m(h)\, n_b\theta_{mb}}{n_s\theta_{ms}R}\left[-1 + F(R) - \frac{1}{R}\int_R^\infty \left[1-F(\pi)\right]d\pi\right]\right\}. \tag{32}$$

As the term in square brackets is negative, $\partial G/\partial R < \beta(h)\left[1-\delta(h)\right] < 1\ \forall R$, so G crosses the 45^o line at most once: if it exists steady state is unique. With $\hat{R}(h)$ denoting steady state as a function of period length, it solves $R = G(R, h)$, which we rearrange as

$$\begin{aligned} & R^2\left[1-\beta(h)-\beta(h)\delta(h)\right] \\ = \ & \beta(h)\left[1-\delta(h)\right]\left\{\left[\rho(h)-\kappa_m(h)\right]R + \frac{\kappa_m(h)\, n_b\theta_{mb}}{n_s\theta_{ms}}\int_R^\infty \left[1-F(\pi)\right]d\pi\right\}. \end{aligned}$$

For any $h > 0$, the LHS is 0 at $R = 0$ and the RHS is strictly positive at $R(h) = 0$. Hence $\hat{R}(h) > 0\ \forall h > 0$. Dividing by h we get

$$\begin{aligned} & R^2\frac{1-\beta(h)-\beta(h)\delta(h)}{h} \\ = \ & \beta(h)\left[1-\delta(h)\right]\left\{\frac{\rho(h)-\kappa_m(h)}{h}R + \frac{\kappa_m(h)}{h}\frac{n_b\theta_{mb}}{n_s\theta_{ms}}\int_R^\infty \left[1-F(\pi)\right]d\pi\right\}. \end{aligned}$$

As $h \to 0$, the LHS approaches $R^2(r+\delta)$ and the RHS approaches $(\rho - \kappa_m) R + \frac{\kappa_m n_b \theta_{mb}}{n_s \theta_{ms}} \int_R^\infty [1 - F(\pi)]\, d\pi$. At $R = 0$, the LHS is 0 and the RHS is strictly positive. Hence, $\hat{R}(0) \neq 0$. Finally, evaluate (32) at $h = 0$ and $\hat{R}(0)$ to get $\lim_{h \to 0} \partial G / \partial \hat{R}_0 = 1$. By the continuity of G, there exists a cutoff $\underline{h} > 0$ such that $\partial G / \partial \hat{R}(h) > -1$ for $h > \underline{h}$, implying a 2-cycle, and $\partial G / \partial \hat{R}(h) \leq -1$ otherwise. As is standard, if a 2-cycle does not exist, no cycles of any order exists. ■

Proof of Claim 1: As discussed in the text there are 4 cases. We consider each in turn.

Case 1: $\gamma = 1$ and $n_s > 0$. For $\gamma = 1$ we need $\pi > \Delta$, which easily reduces to

$$\pi \geq f_1(\rho) \equiv \frac{\rho + \kappa\theta/(1-\theta)}{r + \delta + \alpha\theta} \tag{33}$$

Given $\gamma = 1$, $n = 0$. Finally, $n_s > 0$ reduces

$$\pi \geq \frac{\kappa}{\alpha(1-\theta)} \equiv \pi_0 \tag{34}$$

Case 2: $\gamma = 0$ and $n_s > 0$. Now $\gamma = 0$ requires $\Delta \geq \pi$, which reduces to

$$\pi < \max\{f_2(\rho), f_3(\rho)\} \tag{35}$$

where $f_2(\rho) \equiv \rho / (r + \delta + \alpha\theta)$ and

$$\begin{aligned} f_3(\rho) \quad \equiv \quad & \frac{\delta(1-\theta)(\rho-\kappa) - \kappa r}{2(1-\theta)\delta(r+\delta+\alpha\theta)} \\ & + \frac{\left\{[\delta(1-\theta)(\rho-\kappa) - \kappa r]^2 + 4\delta(1-\theta)(r+\delta+\alpha\theta)\kappa\rho\right\}^{0.5}}{2(1-\theta)\delta(r+\delta+\alpha\theta)}. \end{aligned}$$

Notice f_3 is increasing, concave, $f_3(0) = 0$ and $f_3(\rho_0) = \pi_0$ where π_0 is given in(34). Then $n \geq 0$ reduces to

$$\rho > \frac{\kappa(r+\delta)}{\alpha(1-\theta)} \equiv \rho_0. \tag{36}$$

We also need $n \leq 1$ and $n_s > 0$ but these are redundant given the other conditions.

Case 3: $\gamma \in (0,1)$ and $n_s > 0$. For $\gamma \in (0,1)$ we solve for $\Delta = \pi$ for n and check that $0 \leq n \leq 1$ holds iff $\pi \geq f_1(\rho)$ and $0 < \gamma < 1$ iff $\pi < f_3(\rho)$, which also guarantees $n_s > 0$. Notice that when two pure-strategy steady states exist, as usual, the mixed-strategy steady state does, too.

Case 4: $n_s = 0$. We need $rV_s \leq \kappa$ which reduces to a simple parameter condition. Here obviously $n = 0$ so we only need to check the best response condition for γ, even if it is only relevant off the equilibrium path, since $n_s = 0$ means B never actually gets to decide to consume or store x in equilibrium. For any $\gamma \in [0, 1]$, $\Delta = \rho / (r + \delta)$ as B's trading probability is 0. For $\gamma = 1$, $\Delta < \pi$ iff $\pi > \rho / (r + \delta)$ and $rV_s \leq \kappa$ iff $\pi \leq \pi_0$. For $\gamma = 0$, $\Delta > \pi$ iff $\pi < \rho/(r + \delta)$ and $rV_s \leq \kappa$ iff $\rho \leq \rho_0$. For $\gamma \in (0, 1)$, $\pi = \rho / (r + \delta)$ and $rV_s \leq \kappa$ iff $\pi \leq \pi_0$. Altogether, $n_s = 0$ holds iff $\pi \leq \pi_0$ and $\rho \leq \rho_0$. ■

Proof of Claim 2: Consider the model without M where B decides whether to enter and whether to consume or store x. We have

$$\begin{aligned}
rV_s &= \frac{\alpha (n_b - n)}{1 + n_b} (1 - \theta) \left[\int_0^R \Delta dF(\pi) + \int_R^{\bar{u}} \pi dF(\pi) \right] + \dot{V}_s \\
rV_0 &= \frac{\alpha}{1 + n_b} \theta \left[\int_0^R \Delta dF(\pi) + \int_R^{\bar{u}} \pi dF(\pi) \right] + \dot{V}_0 \\
rV_1 &= \rho - \delta \Delta + \dot{V}_1
\end{aligned}$$

Following the usual procedure, we get

$$\begin{aligned}
\dot{\Delta} &= (r + \delta) \Delta - \rho + \frac{\alpha}{1 + n_b} \theta \left\{ \Delta + \int_\Delta^{\bar{u}} [1 - F(\pi)] \, d\pi \right\} \\
\dot{n} &= -\delta n + \frac{\alpha (n_b - n)}{1 + n_b} F(\Delta) \\
\kappa_b &= \frac{\alpha}{1 + n_b} \theta \left[\int_0^R \Delta dF(\pi) + \int_R^{\bar{u}} \pi dF(\pi) \right]
\end{aligned}$$

Again, using the third equation to eliminate n_b from the others we have a bivariate system

In particular, the entry condition implies

$$\dot{\Delta} = (r + \delta) \Delta - \rho + \kappa_b,$$

which has a unique bounded solution, the steady state, $\Delta = (\rho - \kappa) / (r + \delta)$. Then

$$n_b = \frac{\alpha}{\kappa_b} \theta \left[\int_0^R \Delta dF(\pi) + \int_R^{\bar{u}} \pi dF(\pi) \right] - 1$$

is also a constant, in and out of steady state, while inventories converge to steady state following the $\dot{n}$ equation. Hence there is a unique equilibrium. ■

Proof of Claim 3: Now suppose S decide whether to produce for themselves at cost c and enjoy ρ, or enter the market and produce when they meet B at the same cost c, or not produce. Let V_0 and V_1 be the value functions of S without and with inventory, and V_s the value of S in the market. Let N_s be the total measure of sellers and n_s those in the market. For simplicity, suppose π is degenerate. Then

$$\begin{aligned} V_0 &= \max\{V_1 - c, V_s, 0\} \\ rV_1 &= \rho - \delta(V_1 - V_0) + \dot{V}_1 \\ rV_s &= \frac{\alpha n_b}{n_s + n_b}\theta_s[\pi - c - (V_s - V_0)] + \dot{V}_s \end{aligned}$$

Now we can proceed as in Claim 1 and consider four regimes, although now we do not restrict attention to steady state.

Case 1: All sellers enter. That requires $V_0 = V_s > V_1 - c$ and $n_s = N_s$. There is a unique solution

$$V_s = \frac{\alpha n_b \theta_s}{r(N_s + n_b)}(u - c) \text{ and } V_1 = \frac{\rho + \delta V_s}{r + \delta},$$

and this is an equilibrium for parameters satisfying $u > c$ and

$$\rho \leq f_1(u) \equiv \frac{\alpha n_b \theta_s}{n_b + N_s}u + \left(r + \delta - \frac{\alpha n_b \theta_s}{n_b + N_s}\right)c.$$

Case 2: All sellers produce for themselves. That requires $V_0 = V_1 - c \geq V_s$ and $n_s = 0$. There is again a unique solution

$$V_1 = \frac{\rho - \delta c}{r} \text{ and } V_s = \frac{\alpha \theta_b}{r + \alpha \theta_s}(u - 2c + V_1)$$

and this is an equilibrium for parameters satisfying $\rho \geq f_2(u) \equiv \alpha\theta_s u + (r + \delta - \alpha\theta_s)c$ and $\rho > (r + \delta)c$.

Case 3: Some sellers produce and hold inventory, and others enter the market. That requires $V_0 = V_s = V_1 - c$ and $n_s \in (0, N_s)$. The unique solution is

$$V_1 = \frac{\rho - \delta c}{r}, \; V_s = \frac{\rho - (r + \delta)c}{r}, \text{ and } n_s = \frac{\alpha n_b \theta_s(u - c)}{\rho - (r + \delta)c} - n_b$$

and this is an equilibrium for parameters satisfying $f_1(u) < \rho < f_2(u)$.

Case 4: Sellers do not produce. That requires $V_0 = 0$, $V_1 - c < 0$, $V_s < 0$ and $n_s = 0$. The unique solution is

$$V_1 = \frac{\rho}{r+\delta} \text{ and } V_s = \frac{\alpha\theta_s (u-c)}{r+\alpha\theta_s}$$

and this is an equilibrium for parameters satisfying $\rho < (r+\delta) c$ and $u < c$. This completes all the cases, and implies uniqueness. ■

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