Money and Taxes Implement Optimal Dynamic Mechanisms

Bruno Biais (HEC), Hans Gersbach (ETHZ), Jean-Charles Rochet (TSE),
Ernst Ludwig von Thadden (Mannheim), and Stéphane Villeneuve (TSE)*

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Abstract

We analyze dynamic capital allocation and risk sharing between a principal and many agents, who privately observe their output. Incentive compatibility requires that agents bear part of their idiosyncratic risk. The larger the agents' risk exposure, the larger the rents the principal can extract from them. The optimal mechanism can be implemented as the equilibrium of a market where agents exchange goods for money, needed to pay taxes. Inflation affects agents' portfolio choice between risky capital and safe money. To implement the optimal mechanism, the principal targets an inflation rate such that agents' risk exposure is the same in equilibrium and in the mechanism.

1   Introduction

How should capital be allocated and risks shared in a dynamic production economy without aggregate risk? In the absence of informational frictions, the answer is clear: capital should be allocated according to individual productivities and risks should be eliminated by diversification. However, when information about individual outputs is private, incentive compatibility constraints must be taken into account. This paper studies how these constraints affect capital accumulation and risk sharing.

To address these issues, we consider an infinite horizon economy with a continuum of risk averse agents and a single good that can be consumed or invested as capital, similar to the economy studied in Angeletos [3]. Each agent operates a project whose output is proportional to the amount of capital under her management and subject to idiosyncratic shocks. Individual unit outputs are i.i.d. so that a version of the law of large numbers applies, implying that aggregate output is deterministic.

We assume agents privately observe their individual output and can secretly consume some of it, as in Bolton and Scharfstein [9]. In contrast to output, capital is observable. Applying the revelation principle, we study truthful revelation mechanisms, in which agents truthfully report their output to the principal, who then allocates consumption and capital according to the reports. Thus the dynamic optimal mechanism allocates capital and consumption to maximize the principal's utility, subject to the participation and incentive constraints of the agents and the aggregate resource constraint.

To provide agents with incentives not to divert output, the optimal contract specifies an increase (resp. decrease) of consumption **and** capital for agents whose output is larger (resp. smaller) than expected. Lucky agents (those that perform better in a given period) get more capital to manage in the next period, not because they are more skilled (performance is i.i.d. across agents and across periods) but because this provides incentives to report good performance instead of diverting output. In contrast with the symmetric information case, insurance is imperfect, because full insurance is not incentive compatible. So, the optimal mechanism exposes agents to a fraction of their idiosyncratic risk.

From a mathematical viewpoint, finding the optimal mechanism is challenging, as we need to extend to a continuum of agents the martingale techniques introduced by Sannikov [38] in the one agent case. With only one agent, the Bellman equation that characterizes the optimal mechanism involves the partial derivatives of the value function with respect to two state variables: aggregate capital and the continuation utility promised to the (single) agent by the principal. In contrast, in our model with a continuum of agents, the state variables are aggregate capital and the **entire distribution** of continuation utilities across agents, which belongs to the space of probability distributions over $\mathbb{R}$. So the value function of the principal solves a Bellman equation in an infinite dimensional space. We first determine this Bellman equation, which involves a generalized notion of derivative (the L-derivative[1]) of the value function with respect to the probability distribution of continuation utilities. Then, thanks to our log utility specification, we show that the dimension of states variables can be reduced to two: aggregate capital and the expectation of (a function of) agents' continuation utilities. These are sufficient statistics for the characterization of the optimal mechanism.[2] Thanks to the reduction in the dimension of the state space from infinity to two, we can fully characterize the dynamics of capital and consumption allocations as well as the distribution of continuation utilities across agents.

The optimal direct mechanism is remarkably simple: consumption and capital are allocated among agents proportionally to each agent's equivalent permanent consumption, defined as the constant lifetime stream of consumption giving the agent the same continuation utility as the mechanism. The equivalent permanent consumption of each agent grows at a constant rate in expectation, but is impacted by the agent's performance. The innovation in the growth rate of an agent's consumption or capital is proportional, by a positive constant x, to the agent's idiosyncratic output shock. The proportionally constant x measures the extent to which the agent is exposed to the risk of her idiosyncratic output shock. By raising x the principal relaxes the incentive compatibility condition and can thus extract more rents from the agent, but this reduces allocative efficiency by reducing insurance. Thus there is a rent-efficiency tradeoff. We characterize the set of information-constrained Pareto optimal allocations, which can be parametrized by x. Because agents are exposed to their idiosyncratic shocks, inequality increases over time and agents become more and more heterogenous.[3] Moreover, while aggregate capital and output grow over time, growth is lower than under symmetric information. This is because incentive compatibility constrains how much new capital can be delegated to agents.

This direct revelation mechanism is centralized as all agents report to the principal, who then reallocates consump-

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[1]See Cardaliaguet [11].

[2]Angeletos [3] also avoids the "curse of dimensionality" with a log utility specification. A major difference is that in Angeletos [3] institutions and market incompleteness constraints are exogenous while in our paper they are features of the endogenous optimal mechanism.

[3]More precisely, the coefficient of variation (standard deviation divided by the mean) of continuation utilities across agents increases over time.

tion and capital among them. We show that a more decentralized implementation is possible, in which agents exchange goods against money in a market and the principal intervenes only via money issuance and taxation. When trading in the market, agents face a dynamic portfolio problem à la Merton [35]. They choose how much to invest in capital and money, bearing in mind that the former has higher expected return but is riskier than the latter. The principal influences this portfolio choice by controlling money supply and thus inflation, which affects the excess rate of return of risky capital over money, so that agents' risk exposure is the same in equilibrium as in the optimal mechanism. Different monetary and fiscal policies implement different points on the incentive constrained Pareto frontier. In general the principal uses both taxation and seigneurage to raise revenue and extract rents from the agent.

Our results can be contrasted with the classical welfare theorems. In a convex economy with complete markets and no frictions, all competitive equilibria are efficient (first welfare theorem) and, converserly, all efficient allocations can be decentralized by a competitive equilibrium after appropriate transfers between agents (second welfare theorem). The classical welfare theorems don't apply in our economy with asymmetric information and endogenously incomplete markets. In contrast with the first theorem, competitive equilibria without government intervention are constrained inefficient. However, all constrained optimal allocations can be implemented as a market equilibrium provided the government chooses appropriate monetary and taxation policies. By setting money supply and taxes appropriately, the government can impact individual behaviour to optimally mitigate imperfections. This can be viewed as an extension of the second welfare theorem to an economy with endogenously incomplete markets. An important difference is that, while in the classical second welfare theorem transfers are lump-sum to avoid distorting agents' choices, in our analysis transfers are designed to optimally affect agents' choices.

Literature: Our paper complements several strands of literature.

First, our analysis of dynamic contracting between one principal and many agents is related to the literature analyzing dynamic contracting between one principal and one agent, in particular the seminal work of DeMarzo and Fishman [13], [14] and Sannikov [38], and the following analyses of Biais, Mariotti, Plantin, and Rochet [7], DeMarzo and Sannikov [15], Feng and Westerfield [24], and Di Tella and Sannikov [21]. As in Biais, Mariotti, Rochet, and Villeneuve [8] and DeMarzo, Fishman, He, and Wang [16], firm size is determined by the optimal contract and is useful to provide incentives. However, in contrast to these last two papers, in the present paper there are no capital adjustment costs. This enhances tractability, and gives rise to continuous reallocation of capital. He (2009) offers an interesting alternative approach in which firm size is affected by unobservable agent's effort. This differs from our model in which firm size is directly controlled by the principal, and what is unobservable is output.

The major contribution of the present paper relative to that literature is to embed the contracting problem into a general equilibrium context, with a population of agents and aggregate resource constraints. Thus we shed light on the impact of incentive constraints on the allocation of capital and consumption between agents. In particular, we show how incentive constraints generate increasing inequality in the population. Moreover, we show how the dynamic optimal mechanism can be implemented through a market in which agents trade goods for money, and inequality is regulated by taxing wealth.

Second, our analysis is related to the dynamic macrofinance literature analyzing risk with exogenously incomplete markets (see Bewley [6] , Aiyagari [1], Huggett, 1993, 1997, Krusell and Smith [33], Angeletos, [3], Brunnermeier Sannikov [10], Di Tella [20], and Achdou et al [4].) [4]

The major contribution of the present paper relative to that literature is to provide microfoundations for market incompleteness.[5] Thus, the institutions and constraints we consider are endogenous features of the optimal dynamic mechanism. This helps clarify the consequences of informational frictions. For example, we reconcile two effects which, as explained by Angeletos [3], had so far been viewed as distinct. While the literature in line with Bernanke and Gertler (1989) emphasizes how wealth affects the *ability* to invest in capital, Angeletos [3] emphasizes how wealth affects the *willingness* to hold risky capital. Our mechanism design approach clarifies the common origin of these two forces: incentive compatibility constrains both how much capital agents are allocated and how much of the corresponding idiosyncratic risk they must bear. Consequently, in contrast with Angeletos [3], in our analysis frictions unambiguously lower capital accumulation. Gersbach, Rochet, and Von Thadden [25] rely on the mechanism design analysis of the present paper to study an extension analysing how new public expenditures can be financed in an economy with heterogeneous agents and endogenously incomplete markets.

Third, our focus on money in the implementation of the optimal mechanism links our paper to the new monetarist literature initiated by the seminal papers of Kiyotaki and Wright ([30], [31]) and reviewed in Williamson and Wright

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[4]Our focus on the distribution across agents and our reliance on mean field techniques are in line with Achdou et al [4].

[5]Another difference is that, while most of that literature studies labor income risk, our paper, like Angeletos [3] considers capital return risk.

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[41]. A common theme with that literature is that money arises endogenously, as a useful instrument, instead of being a constraint as in cash in advance models or exogenous as in money-in-the-utility-function models. Money in our implementation encodes the memory of past performance in line with Kocherlakota [32] and provides consumption insurance in line with Berentsen and Rocheteau [5].

There are important differences, however, between our analysis and the new monetarist literature. First, instead of starting from a characterization of optimal allocations in a setting with money, we characterize the optimal mechanism in a real economy with only goods and no money, and then we introduce money as a tool to implement the optimal mechanism. Second, while the new monetarist literature assumes large households (Shi [39]) or the alternation of decentralized and centralized markets (Lagos and Wright, [34]) so that at the beginning of each period all agents start with the same amount of money, in our framework agents have endogenously heterogeneous money holdings, and we characterize the dynamics of this heterogeneity.[6] The third difference is a consequence of the second one: In the new monetarist literature, agents are homogenous at the beginning of each period, so the optimal allocation is pinned down by a static mechanism. In contrast, in our setting agents' continuation utilities vary stochastically over time, so the optimal allocation is set by a dynamic mechanism.[7]

Finally, we complement the mechanism design approach to optimal taxation pioneered by Mirrlees [36], Diamond and Mirrlees [19], and Diamond [17], and further developed by the new dynamic public finance literature (e.g., Golosov, Kocherlakota and Tsyvinski [26], Golosov and Tsyvinski [27], and Farhi and Werning [23]. A major difference is that, in these papers, risk and information asymmetry are about wage earners' skills, while, in our paper, risk and information asymmetry are about managers' capital returns. Correspondingly, unlike in these papers, the dynamic of capital allocation plays a key role in our analysis. Another major difference is that the optimal taxation literature focuses on one policy tool, namely the tax system, while in our set-up, the government chooses also budgetary policy (the consumption of the principal) and monetary policy (how much money is issued).

Structure of the paper: The complete analysis of the continuous time model under asymmetric information is difficult and mathematically complex. In order to build intuition, the next section presents a simple two period model, which illustrates some (but not all) of the economic forces at play in our full model. Then, Section 3 introduces the continuous time model, and solves the symmetric information case, which provides a useful benchmark for the analysis of asymmetric information. In Section 4 we determine the Bellman equation that characterizes the principal value function under asymmetric information. Then we make a guess on the form of the optimal policy, qualitatively close to that obtained under symmetric information, and finally we show that this candidate policy is indeed the full solution of our problem. Section 5 shows that the optimal direct mechanism can be implemented with money and taxes. Section 6 concludes. Proofs not in the text are in Appendix A. Appendix B provides a brief introduction to generalized differential calculus in Wasserstein spaces.

2 The two-period case

To build intuition, we first present a simple version of the model with three dates and no discounting. There is a single good which can be used for consumption or investment. Agents can invest the good and generate returns whose distribution is i.i.d across agents. We first characterize the symmetric first best allocation. Then we turn to the case in which agents privately observe their returns. While the first best allocation is not incentive compatible, we characterize the symmetric second best allocation and show how it can be implemented with money and taxes.

2.1 A simple two-period model

The investment technology has constant returns to scale. There is a benevolent principal[8] and a mass 1 continuum of agents, each endowed with one unit of capital, and identical utility function U, strictly concave, increasing and

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[6]Rocheteau, Weill, and Wong [37] offer an interesting extension of the Lagos and Wright [34] approach in which they characterize the equilibrium distribution of money holdings. But, in line with previous new monetarist analyses, their approach relies on money from the start, in contrast with our approach, which start with a direct mechanism without money and then implement it with money.

[7]Another interesting paper in line with that literature to which our paper is related is Aiyagari and Williamson [2]. In our paper like in theirs a continuum of agents have random outputs and want to share risk, but this is difficult because individual outputs are privately observed. A major difference between their work and ours is that in Aiyagari and Williamson [2] agents don't use capital, while in our analysis agents' ouputs are increasing in the capital they are allocated. Thus, in our paper, unlike in Aiyagari and Williamson [2] capital allocation is key in the provision of incentives, and information asymmetry reduces capital accumulation relative to the first best. Another difference is that Aiyagari and Williamson [2] study the consequences of transportation constraints, which are absent in our framework.

[8]By this we mean that the principal does not consume, and maximizes the welfare of agents. In the continuous time version, the principal is self interested.

differentiable. Time 1 unit returns, which are i.i.d. across agents, can be high ($R^H = 1 + \sigma$) or low ($R^L = 1 - \sigma$) with equal probability $1/2$.[9] To ensure that returns are positive we assume $\sigma < 1$. For simplicity we assume that time 2 returns are deterministic and equal to 1. The second period technology can thus be interpreted as costless storage.

Agents with high return are referred to as type $s = H$ and agents with low return as type $s = L$. After time 1 output is realized, agents consume C_1^s and are allocated capital k_1^s which is stored and consumed at date 2: $C_2^s = k_1^s$. The ex-ante utility of each agent is

$$E\left[U(C_1^s) + U(C_2^s)\right].$$

2.2 The Symmetric First Best Allocation

Since agents are ex ante identical, we focus on the symmetric first best allocation, which is characterized by the consumption profile $\{C_1^s, C_2^s\}$, $s \in \{L, H\}$ maximizing

$$E\left[U(C_1^s) + U(C_2^s)\right],$$

subject to the intertemporal resource constraint:[10]

$$E\left[C_1^s + C_2^s\right] \leq 1.$$

We thus obtain our first proposition:

Proposition 1 *In the symmetric first best allocation, $C_t^s = \frac{1}{2}, \forall (t, s)$.*

The intuition is straightforward. Because returns are i.i.d across agents, the law of large numbers implies that aggregate output is deterministic. Agents are only exposed to idiosyncratic shocks and it is optimal to mutualize these idiosyncratic shocks. Initially there is one unit of good, which can be consumed or invested. Since the rate of return and the discount rate are equal, half the endowment is consumed at time 1 ($C_1^s = \frac{1}{2}$) while the other half is invested at time 1 and consumed at time 2 ($C_2^s = \frac{1}{2}$). Note that this allocation is characterized by complete consumption smoothing over time, and complete insurance against output shocks.

2.3 The Symmetric Second Best Allocation

Next turn to the case in which agents privately observe their time 1 return. By the revelation principle, we can restrict attention to direct mechanisms, where agents report and transfer their output to the principal, who then sets their consumption and capital allocation. When an agent's return is high she can pretend it is low, and secretly consume the difference (2σ). When an agent's return is low, she cannot pretend it is high: to do so she would have to transfer high output to the principal, but such high output is not available to the agent. Thus, there is only one incentive compatibility constraint:

$$U(C_1^H) + U(C_2^H) \geq U(C_1^L + 2\sigma) + U(C_2^L).$$

The incentive compatibility constraint does not hold for the first best allocation, in which $C_1^H = C_2^H = C_1^L = C_2^L$. Maximising agents' expected utility under resource and incentive constraints yields the properties of the second best allocation, stated in the next proposition.

Proposition 2 *In the symmetric second best allocation, consumptions are such that $C_1^H = C_2^H$, $C_2^H > C_2^L$ and $C_1^L > C_2^L$.*

As in the first best allocation, high types have the same consumption in both periods. This is the standard "no distortion at the top" result. By contrast, consumption smoothing is imperfect for low types: $C_1^L > C_2^L$. An agent with high return pretending he had low return secretly consumes an additional 2σ at time 1. This lowers his time 1 marginal utility relative to his time 2 marginal utility. Consequently, giving time 1 consumption to the agent with low returns tightens the incentive constraint less than giving time 2 consumption to that agent. Hence, it is optimal to set $C_1^L > C_2^L$, although this is a distortion relative to the first best. Moreover risk sharing is also imperfect: The intertemporal utility of high types is strictly higher than that of low types. Low consumption after low output, although

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[9]Thus, net average rates of returns are 0, like the discount rate. This is just for the sake of simplicity. In the continuous time analysis below, average rates of returns and discount rates are strictly positive.

[10]The intertemporal resource constraint obtains by adding the time 1 resource constraint $E\left[C_1^s + k_1^s\right] \leq 1$, and the time 2 resource constraint $E\left[C_2^s\right] \leq E[k_1^s]$.

suboptimal under symmetric information, is constrained-optimal under asymmetric information, because it incentivizes truthful reporting. Thus with information asymmetry there is imperfect insurance. Figure 1 plots, for $\sigma = 0.3$ and log utility, the time 1 and time 2 consumptions in the first and second best. In the logarithmic utility case, the second best allocation can be computed (almost) explicitly.

Proposition 3 *When $U(c) = \log c$, the symmetric second best allocation is such that*

$$C_1^H = C_2^H = (\frac{1+\sigma}{2})(1-\mu^2), C_2^L = (\frac{1+\sigma}{2})(1-\mu)^2, C_1^L = (\frac{1+\sigma}{2})(1+\mu)^2 - 2\sigma,$$

where μ is the unique positive solution of the equation: $\mu(1+\mu)^2(1+\sigma) = \sigma(1+3\mu)$. Moreover $\frac{\sigma}{1+\sigma} < \mu < 1$. Correspondingly, in the second best:

1. *Unsuccessful agents are partially insured by successful agents: $(C_2^L + C_1^L) > (C_2^H + C_1^H) - 2\sigma$,*

2. *but informational frictions reduce aggregate investment at $t = 1$ and thus aggregate consumption at date 2: $C_2^H + C_2^L < 1$.*

Property 1 shows that some insurance can be achieved in spite of informational frictions: agents with low output obtain larger consumption in the optimal mechanism than in autarky. But Property 2 shows that informational frictions reduce investment. This will also be the case in the full model presented in the next sections.

Figure 1

Second best and first best in two period model, log utility σ=0.3
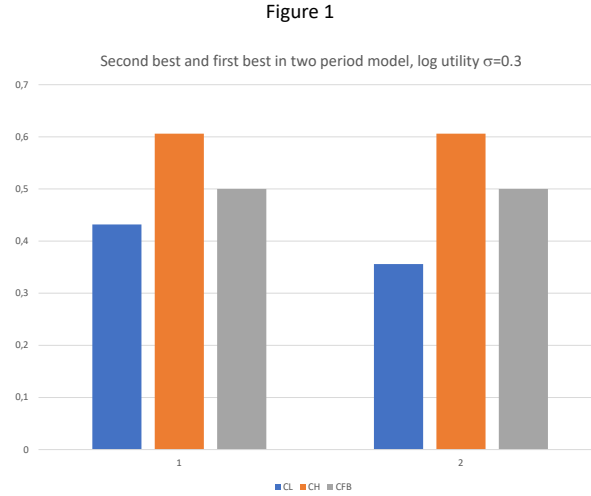
2.4 Implementation with Money and Taxes

The optimal direct mechanism that we have characterized is completely centralized: all agents report to the principal, who then allocates goods across agents. However, a more decentralized implementation of the optimal allocation is possible, in which the good is allocated by a competitive market.[11] In this market the good is traded against fiat money issued by the principal. This money has value because the principal levies taxes at date 1 and requires the agents to

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[11]Note that a pure market solution (no taxes) cannot implement the second best allocation since there are no gains from trade at date 1: successful agents are not willing to transfer resources to unsuccessful ones. In contrast to Diamond Dybvig ([18], if a bond market was created at $t = 1$ it would be inactive, since in our simple two-period model agents have access to a storage technology at $t = 1$.

pay these taxes in money. At date 0 the principal allocates to each agent m_0 units of money, which can be used at time 1 to buy or sell capital k_1^s at price $p > 0$. Note that the principal does not intervene in the good market. Since there is no market at date 2, time 2 consumption is equal to output, $C_2^s = k_1^s$.

At time 1, agent s has R^s units of the good and m_0 units of money. The goods can be used for consumption C_1^s or invested as productive capital k_1^s, while a quantity S^s of goods can be sold for money. If $S^s < 0$, this means the agent is buying goods. So the budget constraint of the agent regarding goods is

$$C_1^s + k_1^s + S^s \leq R^s. \tag{1}$$

After trading, the amount of money held by the agent is equal to her initial endowment (m_0) plus or minus the proceeds of her time 1 sales (S^s). The real wealth of the agent is thus

$$e^s = k_1^s + \frac{m_0}{p} + S^s = \frac{m_0}{p} + R^s - C_1^s.$$

The agent consumes the good and uses her money to pay her taxes. The real value of these taxes is an increasing and differentiable function $\tau(e^s)$ of the agent's wealth. So the budget constraint of the agent regarding money is

$$\tau(\frac{m_0}{p} + R^s - C_1^s) \leq \frac{m_0}{p} + S^s, \tag{2}$$

where p is the price of the good at date 1. This constraint must be binding for all agents, otherwise the value of money would be zero $(p = \infty)$. Fiat money has positive value in this finite horizon model because it is required to pay taxes. At time 1, after observing her type, agent s chooses C_1^s and C_2^s to maximize

$$U(C_1^s) + U(C_2^s),$$

subject to the two constraints (1) and (2), which imply

$$C_1^s + C_2^s + \tau(\frac{m_0}{p} + R^s - C_1^s) \leq R^s + \frac{m_0}{p}.$$

That is expenses, equal to the sum of consumption at both dates and taxes, must be covered by resources, equal to the initial money endowment plus output. We want the solution to the agents' maximization problem to coincide with the constrained optimal allocation. By the Kuhn-Tucker theorem, this is the case when the marginal tax rates faced by each agent satisfy the classical condition:

$$\forall s, \tau'(e^s) = 1 - \frac{U'(C_1^s)}{U'(C_2^s)}, \tag{3}$$

and when their budget constraints are binding:

$$\tau(e^s) = R^s + \frac{m_0}{p} - C_1^s - C_2^s.$$

Feasibility of the optimal allocation then implies that the goods market clears: $\mathbb{E}[S^s] = 0$ and that the real value of the aggregate money stock equals aggregate taxes, i.e.,

$$\frac{m_0}{p} = \mathbb{E}[\tau(e^s)]. \tag{4}$$

We obtain our next proposition.

Proposition 4 *The principal can implement the optimal mechanism $(C_1^s, C_2^s)_{s \in \{L,H\}}$ by distributing an amount m_0 of money to each agent and imposing a non linear wealth tax $\tau(e)$ such that (3) and (4) hold.*

Equation (3) states that the marginal tax rate is the wedge between the intertemporal marginal rate of substitution in the second best and in the first best. We have $\tau'(e^H) = 0$, reflecting that there is no distorsion at the top, and $\tau'(e^L) > 0$ reflecting distortion at the bottom. (4) reflects the equality between money supply and money demand. For given m_0 and τ, p is set so that this market clearing condition holds. The intuition why money and taxes implement the optimal mechanism is the following:

First consider the agents with high time 1 output. They sell some of it, increasing their money holdings, which enables them to pay more taxes. Since taxes are increasing in capital, the ability to pay more taxes translates into the ability to hold more capital. And, since at time 2 agents consume the output from their capital, more capital translates into larger time 2 consumption, which implements the optimal mechanism. This is in line with theories of money as a record of good performance entitling money holders to consumption, i.e., "money as memory" (see Kocherlakota [32]).

Second consider the agents with low time 1 output. They can use some of their money to buy goods, and thus obtain some consumption smoothing. But since they have low money holdings, they cannot afford to pay large taxes, and therefore must have low capital investment and low time 2 consumption, again in line with the optimal mechanism. That unsuccessful agents use money to smooth the impact of shocks on consumption is in line with theories of money as a safe store of value in intertemporal consumption investment settings (see Merton [35], and Berentsen and Rocheteau, [5]).

Taxation allows to create gains from trade between lucky (high types) and unlucky (low types) agents. Since lucky agents want to keep more wealth in order to consume more than unlucky agents at date 2, taxing wealth forces them to sell some of the good to unlucky agents, in order to get more money to pay their taxes. Unlucky agents buy the good because they know they will have to pay less taxes. This allows them to consume more at date 1.

In the infinite horizon analysis below, the above intuitions still hold, but additional effects come into play. For some parameter values money has a bubble component. Moreover, the inflation rate, which is controlled by the principal through money issuance plays an important role in the implementation of the optimal mechanism. Thus, in our continuous time infinite horizon model, there is an optimal level of inflation.

Finally note that, since the horizon is finite, the reason why money has value cannot be that it is a bubble. Here money has value because agents are obliged to pay taxes in money. This is in line with Chartalism (Knapp, 1924) and the general equilibrium literature showing that money has value in finite horizon economies when some agents are obliged to sell something valuable for money (see, e.g., Dubey and Geanakoplos [22]).

3   The infinite horizon case

Although it provides interesting insights, the two period model has important limitations. In particular, there is no incentive compatibility constraint at $t = 1$ to in down the allocation of capital at that time. The intertemporal allocation of capital only becomes important when there are incentive compatibility constraints at different dates. But optimal dynamic contracts are in general difficult to solve, especially in discrete time settings with finite horizons. This is why we now extend the analysis to an infinite horizon model in continuous time. Idiosyncratic shocks are captured by independent Brownian motions, which are easy to define when there is a finite number N of agents, but more tricky with a continuum. We start therefore by describing the model with N agents and then take the limit as N tends to infinity.

3.1   The Model

The principal faces N ex-ante identical agents indexed by $i = 1, ...N$. In order to keep their total mass constant, we assume each of them has mass $1/N$: each agent becomes smaller as their number increases. All agents are infinitely lived with discount rate ρ and logarithmic utility. There is a single good, which can be used for consumption or as capital input in a constant return to scale technology operated by the agents. The total amount of capital K_t is allocated to the agents: agent i invests k_t^i/N units of the good in her production process. The feasibility constraint is

$$K_t = \frac{1}{N} \sum_i k_t^i. \tag{5}$$

The output of agent i is

$$dY_t^i = \frac{k_t^i}{N}[\mu dt + \sigma dB_t^i],$$

where μ is the expected rate of return (net of depreciation) of the technology and B^i, $i = 1, ...N$ are independent Brownian motions, which can be interpreted as idiosyncratic non persistent productivity shocks.

The law of motion of aggregate capital is

$$dK_t = \frac{1}{N} \sum_i \left(k_t^i[\mu dt + \sigma dB_t^i] - c_t^i dt \right) - c_t^P dt, \tag{6}$$

where c_t^i/N is the consumption flow of agent i, while c_t^P is the consumption flow of the principal. (6) is a resource constraint stating that investment (left hand side) is equal to total output net of depreciation minus consumption (right hand side). With a finite number of agents, there is some residual aggregate risk:

$$var(dK_t) = \frac{\sigma^2}{N^2} \sum_i [k_t^i]^2 dt. \tag{7}$$

However when N tends to infinity, if the capital allocation k_t^i is measurable and square Riemann integrable in i, we can determine the limit behavior of the economy. The average amount of capital at date t converges to the Riemann integral of k_t^i:[12]

$$K_t = \int_0^1 k_t^i di, \tag{8}$$

and the law of motion of capital becomes deterministic:

$$dK_t = \left(\mu K_t - \int_0^1 c_t^i di - c_t^P \right) dt, \tag{9}$$

This is because $\frac{1}{N} \sum_i [k_t^i]^2$ has a finite limit ($\int_0^1 (k_t^i)^2 di$) and thus $var(dK_t)$ tends to zero when N goes to infinity.

3.2 Optimal allocations under symmetric information

We first consider the case in which idiosyncratic shocks are observable. This serves as a benchmark to which we then contrast the case in which agents privately observe shocks and can secretly divert output.

3.2.1 The maximization problem

The simplest way to characterize the Pareto frontier of the economy without frictions is to compute the maximum discounted expected utility that the principal can obtain, subject to the resource constraint and the constraint that each agent i gets a given level of utility ω^i.[13] When information is symmetric, since there is no aggregate risk, it is optimal not to expose the agents to any risk. As shown below, this contrasts with the asymmetric information case. Thus, under symmetric information, the consumption of agent i at date t is a deterministic function of t, denoted c_t^i. By construction, it satisfies

$$\omega^i = \int_0^\infty e^{-\rho t} \log c_t^i dt. \tag{10}$$

hereafter referred to as the promise keeping constraint. Similarly the continuation pay-off at date t, satisfies:

$$\omega_t^i = \int_t^\infty e^{-\rho(s-t)} \log c_s^i ds.$$

The objective of the principal is

$$\int_0^\infty e^{-\rho t} \log c_t^P dt, \tag{11}$$

to be maximized subject to the promise keeping condition (10) for all i, and the law of motion of capital:

$$\dot{K}_t = \mu K_t - \int_0^1 c_t^i di - c_t^P. \tag{12}$$

Integrating (12) over time and using the transversality condition ($\lim_{t\to\infty} e^{-\mu t} K_t = 0$), we obtain that the initial amount of capital is equal to the present value of future consumption, discounted at the rate of return on capital:[14]

$$K = \int_0^\infty exp(-\mu t)[\int_0^1 c_t^i di + c_t^P]dt. \tag{13}$$

We restrict attention to anonymous and Markovian stationary controls such that c_t^i and k_t^i only depend on ω_t^i and the state variables K_t and $\mathbb{P}_t$, the probability distribution of ω_t^i. The optimal mechanism maximizes the objective of the principal (11) under the promise keeping constraint (10) and the capital dynamics constraint (13).

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[12]Since the total mass of agents is finite, integrability of $(k_t^i)^2$ implies that (k_t^i) is also integrable.

[13]ω_i can be interpreted as a reservation utility, reflecting an outside option.

[14]This property also held in the simple two period model analyzed above. There the rate of return of capital was equal to 0, so the initial aggregate endowment of capital good was equal to the sum of the future aggregate consumptions.

3.2.2 Characterization of optimal allocations

The next proposition describes the solution of the maximization problem when information is symmetric:

Proposition 5 *Optimal allocations are such that:*

1. *Capital grows at constant rate $\mu - \rho$:*
$$K_t = Ke^{(\mu-\rho)t}.$$

2. *At each date t, the principal consumes a constant fraction γ^P of capital, i.e.,*
$$c_t^P = \gamma^P K_t.$$

3. *Agents' continuation utilities grow linearly:*
$$\omega_t^i = \omega^i + (\frac{\mu - \rho}{\rho})t.$$

4. *At each date t, agent i consumes a constant fraction $\gamma^A = exp[-\frac{\mu-\rho}{\rho}]$ of $exp(\rho\omega_t^i)$, i.e.,*
$$c_t^i = \gamma^A exp(\rho\omega_t^i).$$

5. *For all agents, the ratio $\frac{exp(\rho\omega_t^i)}{K_t}$ is constant over time.*

Property 1 states that aggregate capital grows at a constant rate, equal to productivity μ minus the discount rate ρ. Correspondingly, the flow of aggregate consumption is a fraction ρ of aggregate capital.

Property 2 states that the principal consumes a constant fraction of capital. This arises because the principal has logarithmic utility. Properties 1 and 2, together with (12), imply that the aggregate consumption of the agents is a constant fraction of capital.

Property 3 states that, starting from its initial level ω, an agents' continuation utility grows linearly with time, the trend being equal to the growth rate of capital divided by the discount rate, which is the same for all agents. This implies that inequality across agents does not grow over time, which will not be the case with asymmetric information.

Property 4 states that, at time t, an agent's consumption is a constant fraction of $exp(\rho\omega)$, which can be interpreted as the "equivalent permanent consumption" namely the constant lifetime stream of consumption giving utility ω to an agent. Since the agent's utility function is logarithmic and her discount rate is ρ, the equivalent permanent consumption corresponding to ω is $exp(\rho\omega)$. Combined with properties 1 and 3, it implies that an agent's consumption grows at the same rate as aggregate capital.

This yields Property 5, which states that the ratio of an agent's equivalent permanent consumption to aggregate capital is a constant, equal for all agents, which we denote by z. Aggregating across agents, the ratio of aggregate equivalent permanent consumption to capital is constant and equal to z:
$$\frac{\int exp(\rho\omega_t)d\mathbb{P}(\omega)}{K_t} = z.$$

We can now compute the value function of the principal:
$$V = \int_0^\infty e^{-\rho t} \log c_t^P dt, \tag{14}$$

The above proposition implies that this value function only depends on two state variables: aggregate capital K and z, which summarizes all the necessary information on the probability distribution $\mathbb{P}$ of ω. This reduces the dimensionality of the problem from ∞ to 2. The value function of the principal can be computed explicitly:
$$\rho V = log(\rho exp\frac{\mu - \rho}{\rho}K - zK) \tag{15}$$

The first term in the log on the right hand side of (15) is the total amount of constant certainty equivalent consumption that can be allocated among the principal and the agents. It represents the present value of consuming a fraction ρ of capital K growing at rate $\mu - \rho$. The second term in the log on the right hand side of (15) is the aggregate equivalent

permanent consumption $\int \exp \rho \omega d\mathbb{P}(\omega) = zK$ of the agents, which cannot exceed $[\rho exp \frac{\mu - \rho}{\rho}]K$. Thus the value function of the principal can be written

$$V(K, z) = \frac{\log K}{\rho} + v(z),$$

where $\rho v(z) = \log(\rho exp \frac{\mu - \rho}{\rho} - z)$ is only defined for z in a bounded interval: $0 \le z \le \rho exp \frac{\mu - \rho}{\rho}$. Similar properties will also hold in the asymmetric information case. Finally, the Pareto frontier is linear in the space of equivalent permanent consumptions:

$$exp(\rho V) + \int exp(\rho \omega)d\mathbb{P}(\omega) = [\rho exp \frac{\mu - \rho}{\rho}]K, \tag{16}$$

where the left-hand side is the sum of the principal's equivalent permanent consumption and the aggregate agents' permanent consumption, while the right-hand side is the total amount of equivalent permanent consumption to be allocated among the principal and the agents. It reflects that the total surplus ($\rho exp[\frac{\mu - \rho}{\rho}]$) must be shared between the principal and the agents. The Pareto frontier is depicted in Figure 2.

Figure 2

Certainty equivalents and Pareto frontier
σ = 0.3, ρ = 0.01, μ = 0.05
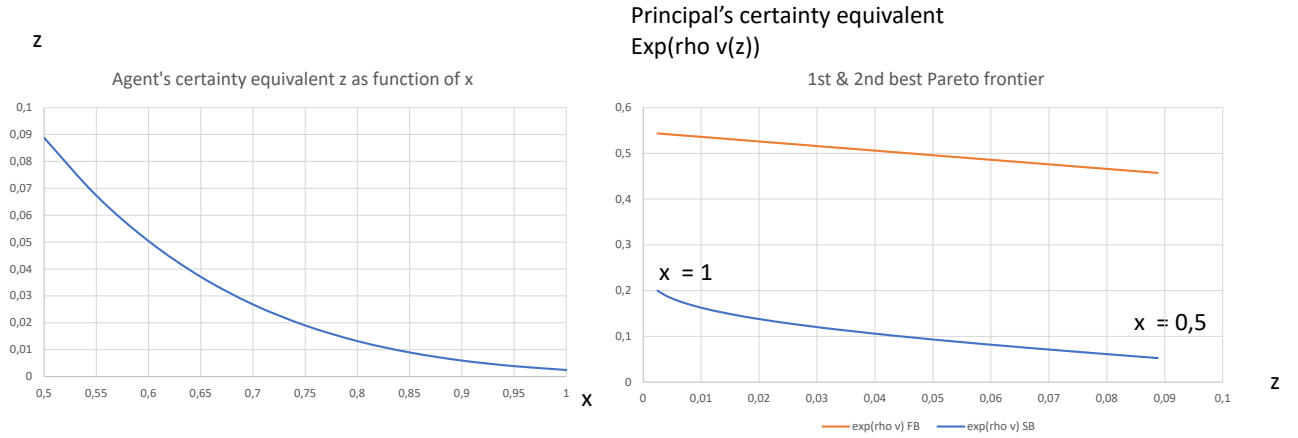
4 Optimal allocations under asymmetric information

We now turn to the case in which agents privately observe their individual output. By the revelation principle, we consider revelation mechanisms. A mechanism is a mapping from the realized output dY_t^i, reported and delivered by agent i to the principal, into consumption and capital allocations for the agent. Since agents privately observe output, they can be tempted to divert a part of it and secretly consume it. To avoid this, the mechanism must induce truthful revelation, i.e., it must be incentive compatible.

4.1 Incentive compatibility

Consider an agent who would want to divert resources and consume secretly. Assuming the agent can only make absolutely continuous changes in the output process, the amount diverted is denoted by $\delta_t \, dt$. Defining

$$d\hat{B}_t^i = dB_t^i - \delta_t \, dt, \tag{17}$$

the dynamics of reported output writes as

$$d\hat{Y}_t^i = \mu k_t^i \, dt + \sigma k_t^i d\hat{B}_t^i.$$

Since the agent cannot secretly store, diversion cannot be negative: $\delta_t \geq 0$ for every t. The time 0 expected utility of an agent i who adopts a diversion strategy δ_t is

$$\omega_0^i = \sup_\delta \ \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log(c_t^i + \sigma k_t^i \delta_t) dt\right].$$

To provide incentives for truthful revelation, the principal changes the continuation utility of the agent as a function of her reports. Hence, by the martingale representation theorem, the dynamics of the continuation utility of agent i is

$$d\omega_t^i = (\rho \omega_t^i - \log(c_t^i)) \, dt + \sigma y_t^i d\hat{B}_t^i, \tag{18}$$

where y_t^i is a B_t^i-adapted process. On the equilibrium path we have

$$d\omega_t^i = (\rho \omega_t^i - \log(c_t^i)) \, dt + \sigma y_t^i dB_t^i. \tag{19}$$

Intuitively, y_t^i is the sensitivity of the agent's continuation utility with respect to her report. The principal must choose this sensitivity to incentivize the agent to report her output truthfully. The state variable for agent i is her continuation utility ω_t^i, so instead of denoting her consumption by c_t^i, we hereafter denote it by $c_t^A(\omega_t^i)$. An intuitive examination of the incentive compatibility condition is the following. The incentive compatibility condition states that the agent must be better off revealing dB_t truthfully, and getting

$$\log(c_t^A(\omega_t))dt + \sigma y_t(\omega_t)dB_t$$

than underreporting: $d\hat{B}_t = dB_t - \delta dt$ and getting

$$\log(c_t^A(\omega_t) + \sigma \delta k_t(\omega_t))dt + \sigma y_t(\omega_t)d\hat{B}_t$$
$$= \log(c_t^A(\omega_t) + \sigma \delta k_t(\omega_t))dt + \sigma y_t(\omega_t)(dB_t - \delta dt).$$

So the incentive compatibility condition is

$$\sigma y_t \geq \sup_{\delta \geq 0} \frac{\log((c_t^A(\omega_t) + \sigma \delta k_t(\omega_t)) - \log(c_t^A(\omega_t))}{\delta} = \frac{\sigma k_t(\omega_t)}{c_t^A(\omega_t)}.$$

This means that the sensitivity of continuation utility to performance has to be larger than the product of the capital $k_t(\omega_t)$ managed by the agent by her marginal utility of consumption, i.e.,

$$y_t(\omega_t) \geq \frac{k_t(\omega_t)}{c_t^A(\omega_t)}.$$

This leads to our next proposition.

Proposition 6 *The incentive compatibility condition is equivalent to the inequality*

$$\forall t, y_t(\omega_t) \geq \frac{k_t(\omega_t)}{c_t^A(\omega_t)}. \tag{20}$$

The incentive compatibility condition (20) implies that, in contrast with the symmetric information case, agents cannot fully mutualize the risk of their idiosyncratic shocks. Condition (20) also shows there is a tradeoff between risk-sharing and investment: providing more insurance to the agent, by reducing the sensitivity of her continuation value to output shocks is possible only at the cost of reducing capital relative to consumption. This is because increasing capital, and therefore output, increases the amount of resources the agent can divert, which tightens the incentive constraint.

This tradeoff is similar to that arising in Biais, Mariotti, Rochet and Villeneuve [8], where size of operation (similar to capital in the present context) was limited by incentive compatibility.

The agents being risk averse, it is never optimal for the principal to expose them to more risk than required by the incentive compatibility condition. In other words the incentive constraint (20) is always binding and we can eliminate the capital allocation variable by writing $k_t^i = y_t^i c_t^i$. Since $\mu > 0$, it is optimal to fully allocate the capital stock to the agents, implying that the aggregate capital constraint (8) writes as

$$\int y_t(\omega_t)c_t^A(\omega_t)d\mathbb{P}(\omega_t) = K_t$$

4.2   The Hamilton-Jacobi-Bellman equation

As in the first best, the value function of the principal does not depend on the specific value function of each individual, but on the distribution $\mathbb{P}$ of agents' continuation pay offs. It also does not depend on individual outputs or capital, but on aggregate capital, which is deterministic, and on aggregate output which is linear in aggregate capital. So K and $\mathbb{P}$ are the state variables of the principal's maximization problem. That is, the principal problem is a deterministic control problem in a space that is the product of $\mathbb{R}$ by the space $\mathcal{P}_2(\mathbb{R})$ of probability measures on $\mathbb{R}$ with a finite second moment, which we endow with the Wasserstein distance (see for example the standard references [12], [11] and [40] and Appendix B). We characterize the principal's value function as the unique solution to the dynamic programming Hamilton-Jacobi-Bellman (HJB) equation in that space.

The main difficulty for exploiting the dynamic programming principle is to differentiate functionals defined on the Wasserstein space. There are various notions of derivatives with respect to measures which have been developed in connection with the theory of optimal transport and using Wasserstein metric on the space of probability measures, for details see Villani (2009) and Appendix B of the present paper. For our purpose, we use the notion of L-differentiability that is presented in appendix B. Following the traditional approach for control problems, we first determine the shape of the HJB equation that the value function of the principal must satisfy (necessary condition) and then establish a verification theorem showing that regular solutions of this HJB equation solve our control problem (sufficient condition). To do so, consider the control problem of the principal

$$V(K,\mathbb{P}) = \sup_{(c_t^A(.),c_t^P,y_t(.))\in\mathcal{K}} \int_0^\infty e^{-\rho t}\log c_t^P dt, \tag{21}$$

where the state equations are given by

$$\dot{K}_t = \mu K_t - c_t^P - \int c_t^A(\omega)d\mathbb{P}(\omega), \tag{22}$$

$$d\omega_t = [\rho\omega_t - \log c_t^A(\omega)]dt + \sigma y_t dB_t, \tag{23}$$

and where the supremum is taken over the set $\mathcal{K}$ of admissible Markov controls (c^A, c^P, y) such that

$$\int y_t(\omega)c_t^A(\omega)d\mathbb{P}(\omega) = K_t, \tag{24}$$

where we observe that the process K_t is deterministic.

A second difficulty is that this control problem involves a constraint (24) that mixes control variables and state variables. To deal with this constraint, we introduce a related, unconstrained, problem as follows: for each function λ (defined on the product of $\mathbb{R}$ by the space $\mathcal{P}_2(\mathbb{R})$ of probability measures on $\mathbb{R}$), which we will call from now on the *Lagrange multiplier*, consider the control problem

$$V_\lambda = \sup_{(c^A,c^P,y)} \int_0^\infty e^{-\rho t}\left(\log c_t^P + \lambda(K_t,\mathbb{P})\left(K_t - \int y_t(\omega)c_t^A(\omega)d\mathbb{P}(\omega)\right)\right)dt.$$

We first state a result that establishes a link between the principal's value V and V_λ.

Proposition 7 *Suppose that for every Lagrange multiplier process, one can find an optimal control $u_\lambda = (c_\lambda^A, c_\lambda^P, y_\lambda)$ such that*

$$V_\lambda = \int_0^\infty e^{-\rho t}\left(\log c_{\lambda,t}^P + \lambda(K_t,\mathbb{P})\left(K_t - \int y_{\lambda,t}(\omega)c_{\lambda,t}^A(\omega)d\mathbb{P}(\omega)\right)\right)dt.$$

Moreover, suppose that there exists $\lambda_0(.)$ such that $K_t = \int y_{\lambda_0,t}(\omega)c_{\lambda_0,t}^A(\omega)d\mathbb{P}(\omega)$, i.e. $u_{\lambda_0} \in \mathcal{K}$. Then, $V = V_{\lambda_0}$ and u_{λ_0} solves the constrained principal problem.

We are now in a position to derive the HJB equation associated with the unconstrained problem.

Proposition 8 *If the value function of the principal is sufficiently regular,[15] it satisfies the following HJB equation:*

$$\rho V(K, \mathbb{P}) = \sup_{c^A(),c^P,y()} \left\{ \log c^P + \lambda(K, \mathbb{P}) \left(K - \int c^A(\omega)y(\omega)d\mathbb{P}(\omega) \right) \right. \tag{25}$$

$$+ V_K(K, \mathbb{P}) \left(\mu K - c^P - \int c^A(\omega)d\mathbb{P}(\omega) \right)$$

$$\left. + \int \partial_\omega \delta V[K, \mathbb{P}](\omega)(\rho\omega - \log c^A(\omega))d\mathbb{P}(\omega) + \int \partial_{\omega\omega}\delta V[K, \mathbb{P}](\omega)\frac{\sigma^2}{2}y^2(\omega)d\mathbb{P}(\omega) \right\},$$

where δV denotes the L-gradient of V with respect to the measure $\mathbb{P}$ and ∂_ω (respectively $\partial_{\omega\omega}$) denote its first (respectively second) partial derivative in ω, while λ denotes the Lagrange multiplier associated with the capital allocation constraint.

Inspired by classical verification theorems for stochastic control of diffusion processes, we prove the following result, which is a consequence of the Itô formula given in Appendix B for functions defined on the Wasserstein space.

Proposition 9 *(Verification Theorem) Let $\lambda(.)$ be a Lagrange multiplier, and $v^\lambda(K, \mathbb{P})$ be C^1 with respect to K and C^2 with respect to mathbbP. Suppose that v^λ is a solution to (25) with the transversality condition $\lim_{t\to+\infty} e^{-\rho t}v^\lambda(K_t, P_{\omega_t^\mu}) = 0$ and there exists a control u_λ^* attaining the maximum in (25). Then $v^\lambda = V_\lambda$. Moreover, if there is a Lagrange multiplier λ_0 such that $u_{\lambda_0}^* \in \mathcal{K}$ then $v^{\lambda_0} = V$.*

4.3 A guess-and-verify approach

We now guess the form of the solution to the optimal control problem and show that the corresponding value function satisfies the Hamilton-Jacobi-Bellman equation (25), so that the guess is the actual solution of the problem.

4.3.1 A restricted control problem

Guided by the characterization of first-best allocations, we conjecture that optimal controls satisfy

$$C_t^P = \gamma^P K_t, C_t^A(\omega) = \gamma^A \exp(\rho\omega_t), \tag{26}$$

where γ^P and γ^A are positive constants. We also posit that $y_t(\omega) \equiv y$ is constant. This is what we call the restricted principal's problem. In the restricted problem, the feasibility constraint (24) gives for all $t \geq 0$,

$$K_t = y\gamma^A \int \exp(\rho\omega_t)d\mathbb{P}(\omega) = y\gamma^A Z_t,$$

where

$$Z_t = \mathbb{E}[\exp(\rho\omega_t)]. \tag{27}$$

As a consequence, the ratio $\frac{Z_t}{K_t} \equiv z$ must be constant, and γ^A must be equal to the inverse of yz. Substituting (26) into (22) and using $\gamma^A = 1/(yz)$ and (27), we obtain the growth rate of capital

$$g := \mu - \gamma^P - \frac{1}{y}. \tag{28}$$

Since the ratio $\frac{Z_t}{K_t}$ is constant, the growth rates of K_t and Z_t must be equal. Thus

$$\frac{dZ_t}{Z_t} = \mathbb{E}[\rho d\omega_t] + \frac{\rho^2\sigma^2 y^2}{2}dt = gdt,$$

which implies the constraint:

$$\mu - \gamma^P - \frac{1}{y} = -\rho log\gamma^A + \frac{\rho^2\sigma^2 y^2}{2}.$$

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[15]By this we mean that it is differentiable in K, L-differentiable in $\mathbb{P}$ (see Appendix B) and that its L-gradient in $\mathbb{P}$ is twice differentiable with respect to ω.

Thus the value function of the restricted problem can be computed as

$$V(K, \mathbb{P}) = \frac{\log K}{\rho} + v(z). \tag{29}$$

where the function $v(z)$ satisfies

$$\rho v(z) = \sup_{y, \gamma^P} [\log \gamma^P + \frac{\mu - \gamma^P - \frac{1}{y}}{\rho}], \tag{30}$$

under the constraint:

$$\mu - \gamma^P - \frac{1}{y} = \rho \log yz + \frac{\rho^2 \sigma^2 y^2}{2}. \tag{31}$$

Feasibility requires that $\gamma^P > 0$, and thus that there exists $y > 0$ such that

$$\mu - \rho \log z > \frac{1}{y} + \rho \log y + \frac{\rho^2 \sigma^2 y^2}{2}.$$

This means that z must be smaller than $z_{\max}$, defined by

$$z_{\max} := \max_y \frac{1}{y} exp[\frac{\mu}{\rho} - \frac{1}{\rho y} - \frac{\rho \sigma^2 y^2}{2}].$$

Substituting γ^P from (30) into (31), we obtain the next proposition, whose proof is in the appendix:

Proposition 10 *Let* $z = \frac{\int \exp(\rho \omega) d\mathbb{P}(\omega)}{K}$. *For* $0 < z < z_{max}$, *the value function of the restricted principal's problem writes as*

$$V(K, \mathbb{P}) = \frac{\log K}{\rho} + v(z). \tag{32}$$

where the function $v(z)$ *satisfies*

$$\rho v(z) = \sup_y [\log(\mu - \frac{1}{y} - \rho \log yz - \frac{\rho^2 \sigma^2 y^2}{2}) + \log yz + \frac{\rho \sigma^2}{2} y^2]. \tag{33}$$

The solution to this problem is denoted $y(z)$. *The corresponding propensities to consume are*

$$\gamma^P(z) = \rho - \frac{1}{y(z) + \rho \sigma^2 y(z)^3}, \tag{34}$$

for the principal and $\gamma^A(z) = \frac{1}{zy(z)}$ *for the agent.*

In line with the incentive compatibility condition (20), which implies that y must be strictly positive as long as agents hold strictly positive capital, inspection of (32) reveals that the solution of the restricted principal's problem involves $y > 0$: in the optimal allocation, agents must bear some of their idiosyncratic risk.

4.3.2 The general case

We now show that the value function of the restricted problem satisfies the Bellman equation (25) and thus solves the complete problem. To do so, we substitute $V(K, \mathbb{P})$ from (32) in the HJB equation (25). We first compute the partial derivatives of order one:

$$V_K = \frac{1 - \rho z v'(z)}{\rho K}, \delta V = \rho exp(\rho \omega) \frac{v'(z)}{K}$$

and then the L-derivatives of V are

$$\partial_\omega(\delta V) = \rho \delta V, \partial_{\omega \omega}(\delta V) = \rho^2 \delta V$$

The Bellman equation becomes

$$\log K + \rho v(z) = \sup[log(\gamma^P K) + \lambda(K - \int \gamma^A(\omega) y(\omega) exp(\rho \omega) d\mathbb{P})$$

$$+ [\frac{1}{\rho} - z v'(z)][\mu - \gamma_P - \frac{\int \gamma^A(\omega) exp(\rho \omega) d\mathbb{P}}{K}] + v'(z) \int \rho exp(\rho \omega)(- \log \gamma^A(\omega) + \frac{\rho \sigma^2}{2} y^2(\omega)) d\mathbb{P}(\omega)].$$

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Note that all the terms involving $\gamma^A(\omega)$ and $y(\omega)$ are multiplied by the same function of ω, namely the product of $exp(\rho\omega)$ by the density of $\mathbb{P}(\omega)$. Thus the pointwise maximum is attained for the same couple (y, γ^A), independently of ω. This implies that the solution is the same as that of the restricted problem, where we have assumed y and γ^A constant. Thus we can replace γ^A by $\frac{1}{yz}$ and γ^P by $(\mu - \frac{1}{y} - \rho \log yz - \frac{\rho^2\sigma^2 y^2}{2})$ and the Bellman equation simplifies into:

$$\rho v(z) = \sup_{y} [\log(\mu - \frac{1}{y} - \rho \log yz - \frac{\rho^2\sigma^2 y^2}{2}) + \log yz + \frac{\rho\sigma^2}{2}y^2],$$

which is the definition of the function $v(z)$. Thus we have established that the value function in (32) satisfies the Bellman equation of the full problem, which is the main result of our paper:

Proposition 11 *The value function of the full problem is*

$$V(K, \mathbb{P}) = \frac{logK}{\rho} + v(z),$$

where $z = \frac{\int exp(\rho\omega)d\mathbb{P}(\omega)}{K}$) and the function v is defined by equation (33). The solution is such that:

$$k(\omega) = \frac{exp(\rho\omega)}{z}, C^A(\omega) = \frac{exp(\rho\omega)}{zy(z)}, \gamma^P = \rho - \frac{1}{y(z) + \rho\sigma^2 y^3(z)},$$

where $y(z)$ is defined in Proposition (10).

4.4 Properties of second best allocations

Taking stock of the analysis above, the next proposition summarizes the properties of optimal information constrained allocations. These properties are drastically simplified by the fact that date t allocations only depend on two state variables, namely the capital stock K_t and the ratio $z_t \equiv \frac{\int exp(\rho\omega_t)d\mathbb{P}(\omega)}{K_t}$. Moreover, along the optimal trajectories, this ratio is constant over time: $z_t \equiv z$, and optimal controls can all be expressed as functions of $y = y(z)$ defined in Proposition (10).

Proposition 12 *Second best optimal allocations are such that:*

1. *Capital grows at a constant rate*

$$g = \mu - \rho - \frac{\rho\sigma^2 y}{1 + \rho\sigma^2 y^2}, \tag{35}$$

   which is lower that the first best growth rate $\mu - \rho$.

2. *Agents' continuation utilities follow a drifted Brownian motion:*

$$\omega_t = \omega + \left(\frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2}\right) t + \sigma y B_t. \tag{36}$$

3. *At each date t, the principal consumes a constant fraction of the capital stock: $C_t^P = \gamma^P K_t$, where*

$$\gamma^P = \rho - \frac{1}{y + \rho\sigma^2 y^3}. \tag{37}$$

4. *At each date t, an agent consumes a constant fraction of $exp(\rho\omega_t)$: $C_t^A(\omega) = \gamma^A exp(\rho\omega_t)$, where*

$$\gamma^A = exp[-\frac{\mu - \rho}{\rho} + \frac{\rho^2\sigma^2 y}{1 + \rho\sigma^2 y^2} + \frac{\rho^2\sigma^2 y^2}{2}].$$

Property 1 shows that frictions reduce growth. This reflects incentive constraints, which restrict investment. When $\sigma = 0$, there is no incentive problem and the growth rate is equal to its first best level.

Property 2 implies that the cross section of agents' continuation payoffs gets more dispersed as time goes by. Even if all agents are ex ante identical, inequality necessarily increases over time, due to incentive compatibility constraints. Moreover, there is a simple relation between the continuation utility of an agent at date t and its performance over

$(0, t)$. Indeed, the average productivity of the agent over $(0, t)$ is just $\mu + \sigma \frac{B_t}{t}$. Optimal compensation implies a simple, affine, relation between the continution utility ω_t and this performance measure, similarly to Holmstrom and Milgrom [29].

Finally, Properties 3 and 4 are similar to the first best case. This simplicity is due to our assumption that utilities are logarithmic and aggregate productivity is constant. A characterization of optimal second best allocations in more general cases is probably much more difficult.

The above properties are parametrized by the sensitivity of agent's continuation utility to performance, y. Varying y does not qualitatively alter these properties, but it generates quantitative changes, e.g., in growth rates or principal's share of consumption. Below, we show how the information constrained Pareto frontier can be written as a function of y.

4.5 Information Constrained Pareto Frontier

The above analysis yields a characterization of the information constrained Pareto frontier in the space of equivalent permanent consumptions. To facilitate its representation, we focus on the case in which all agents start with the same continuation pay-off ω. We also take $K = 1$. In this case, $V(K, \mathbb{P})$ in (32) simplifies to $v(\exp(\rho\omega))$. The continuation utility of the agent is

$$\omega = E[\int_0^\infty e^{-\rho t} \log(C_t^A) dt] = \frac{\log \frac{1}{y}}{\rho} + \frac{\mu - \gamma^P - \frac{1}{y}}{\rho^2} - \frac{\sigma^2 y^2}{2},$$

while that of the principal is

$$v(\exp(\rho\omega)) = \int_0^\infty e^{-\rho t} \log(C_t^P) dt = \frac{\log \gamma^P}{\rho} + \frac{\mu - \gamma^P - \frac{1}{y}}{\rho^2}.$$

Substituting γ^P from (34) into ω and $v(\omega)$ enables us to parameterize the Pareto frontier as a function of y alone. We obtain that the equivalent permanent consumption of the agent is

$$\exp(\rho\omega) = \frac{1}{y} exp[\frac{g}{\rho} - \frac{\rho\sigma^2 y^2}{2}], \tag{38}$$

where g is the growth rate given in (35), while the equivalent permanent consumption of the principal is

$$\exp \rho v(\exp(\rho\omega)) = (\rho - \frac{1}{y + \rho\sigma^2 y^3}) exp[\frac{g}{\rho}]. \tag{39}$$

(38) reflects that each agent consumes a fraction $\frac{1}{y}$ of its capital under management, which grows at average rate g, with volatility σy generating a risk premium, and is discounted at rate ρ. Similarly (39) reflects that the principal consumes a fraction $(\rho - \frac{1}{y + \rho\sigma^2 y^3})$ of the capital stock, but is not impacted by any risk, so that unlike in (38) there is no risk premium. As mentioned above, when $\sigma = 0$ there is no incentive problem. Correspondingly (38) and (39) reduce to

$$exp(\rho\omega) + exp(\rho v(\exp(\rho\omega))) = \rho exp[\frac{\mu - \rho}{\rho}],$$

the equation of the first best Pareto frontier (16), evaluated in the case in which all agents have the same utility ω and $K = 1$.

5 Implementation by money and taxes

The direct revelation mechanism characterized above is completely centralized: all agents report to the principal, who then reallocates goods among agents. We now show that a more decentralized implementation is possible, in which the allocation of goods results from the equilibrium of a competitive market. In that implementation, the principal does not intervene in the reallocation of goods among agents, and relies only on monetary policy (affecting the inflation rate π) and fiscal policy (affecting the tax rate τ.)

Our analysis proceeds in two steps. First, we characterize the equilibrium allocation arising for a given policy (π, τ). There we show how the choice of π and τ determines the agents' and principal's equilibrium consumption processes, as well as the equilibrium growth of output and money supply. Second, we show that, for any second best allocation,

i.e., agents' and principal's second best consumption processes, there exists a policy (π, τ) for which this allocation corresponds to the only rational expectations equilibrium where the price of the good is finite, i.e., money has value.[16] Thus, we obtain a form of second welfare theorem.

5.1 Equilibrium

Our equilibrium analysis proceeds in three steps. First, we characterize the optimal consumption and investment of an agent for a given public policy (π, τ). Second, we spell out the market clearing condition, stating that, at each point in time, the supply of goods is equal to the demand for goods. Third, we derive the equilibrium growth rate induced by policy (τ, π) for output and money supply, as well as the equilibrium consumption share of the principal.

5.1.1 Agent's optimal policy

At $t = 0$, the principal endows each agent with money m_0, targets a constant inflation rate π and announces a constant tax rate τ. Normalizing p_0 to 1, the price of the good in money at time t is $p_t = \exp(\pi t)$. Agents hold capital (k_t) and money (m_t), so an agent's real wealth at time t is

$$e_t = k_t + \frac{m_t}{p_t}. \tag{40}$$

The dynamics of the capital holdings k_t of a given agent is given by:

$$dk_t = k_t(\mu dt + \sigma dB_t) - c_t dt - ds_t, \tag{41}$$

where ds_t denotes the agent's sales (purchases if negative) on the good market. Similarly, the dynamics of the agent's real money balances are

$$d(\frac{m_t}{p_t}) = ds_t - (\pi \frac{m_t}{p_t} + \tau e_t)dt, \tag{42}$$

Adding (41) and (42), ds_t cancels out and we obtain the dynamics of the agent's wealth

$$de_t = k_t(\mu dt + \sigma dB_t) - [c_t + \tau e_t + \pi(e_t - k_t)]dt. \tag{43}$$

Since there are no transaction costs, the agent can costlessly continuously rebalance her portfolio of money and capital and the only constraint is the wealth constraint. So e_t is the agent's state variable, while k_t and c_t can be viewed as the control variables. Equation (43) shows that the change in wealth of an agent is equal to output, minus consumption, taxes, and the decline in the real value of money holdings due to inflation. The latter can be interpreted as an inflation tax. Equation (43) and Ito's lemma imply that the value function $u(e)$ of the agents satisfies the following Bellman equation

$$\rho u(e) = Max_{k,c}[log c + u'(e)[\mu k - c - \tau e - \pi(e - k)] + \frac{\sigma^2 k^2}{2}u''(e)]. \tag{44}$$

The first order condition with respect to c is

$$\frac{1}{c} = u'(e).$$

The first order condition with respect to k is

$$k = \frac{\mu + \pi}{-\frac{u''(e)}{u'(e)}\sigma^2}.$$

Homogeneity implies that the value function is an affine transformation of $\log(e)$:

$$u(e) = \frac{\log(e)}{\rho} + u(1), \tag{45}$$

which implies

$$u'(e) = \frac{1}{\rho e}, u''(e) = -\frac{1}{\rho e^2}. \tag{46}$$

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[16]There is also a no trade equilibrium where money has no value, leading to the autarchic allocation, which is not constrained optimal. The principal can eliminate this bad equilibrium by allowing convertibility of money in the good for a unit price. This convertibility is only needed at date 0 in order to "anchor" expectations.

So the first order conditions yield

$$c = \rho e, \tag{47}$$

and

$$k = \frac{\mu + \pi}{\sigma^2} e. \tag{48}$$

That consumption and capital are constant fractions of wealth stems from the logarithmic utility specification. Denoting

$$x := \frac{\mu + \pi}{\sigma^2}, \tag{49}$$

the optimal portfolio choice of the agent is to invest a fraction x of her wealth in the risky asset and a fraction $1 - x$ in money, the safe asset. Condition (49) shows that the fraction of her wealth an agents invests in the risky asset is increasing in the inflation rate π, which determines the rate of return on money holdings.

5.1.2 Market clearing

Market clearing requires that the aggregate supply of goods by the agent be equal to the consumption of goods by the principal

$$\mathbb{E}[ds_t] = c_t^P dt. \tag{50}$$

First, consider the left-hand side of (50). Since optimality requires a constant ratio of capital to wealth, each agent must buy or sell capital to equalize the growth rate of capital to that of wealth:

$$\frac{dk_t}{k_t} \equiv \frac{de_t}{e_t}.$$

The dynamics of an agent's capital holdings (41), combined with $c_t = \rho e_t$ and $k_t = x e_t$ imply

$$\frac{dk_t}{k_t} = (\mu dt + \sigma dB_t) - \frac{\rho}{x} dt - \frac{ds_t}{k_t}. \tag{51}$$

The dynamics of an agent's wealth, (43), combined with $c_t = \rho e_t$ and $k_t = x e_t$ imply

$$\frac{de_t}{e_t} = x(\mu dt + \sigma dB_t) - [\rho + \tau + \pi(1 - x)]dt. \tag{52}$$

Equating the two yields

$$\frac{ds_t}{k_t} = (1 - x)(\mu dt + \sigma dB_t) - \frac{\rho}{x} dt + [\rho + \tau + \pi(1 - x)]dt,$$

which determines individual good sales ds_t

$$ds_t = [(\mu - \frac{\rho}{x} + \pi)(1 - x) + \tau]k_t dt + \sigma(1 - x)k_t dB_t,$$

and aggregate sales

$$\mathbb{E}[ds_t] = \left[(\mu - \frac{\rho}{x} + \pi)(1 - x) + \tau\right] K_t dt. \tag{53}$$

Second, turn to the right-hand side of (50), i.e., the consumption of the principal. By the budget constraint of the principal, this consumption is equal to the sum of seigneurage and fiscal revenues, that is

$$c_t^P = \mathbb{E}[g_M \frac{m_t}{p_t} di + \tau e_t], \tag{54}$$

where g_M is the growth rate of the money supply. Now, by (40) and (48)

$$\frac{m_t}{p_t} = e_t - k_t = k_t \frac{1 - x}{x}.$$

Substituting in (54) we have

$$c_t^P = \left(g_M \frac{1 - x}{x} + \frac{\tau}{x}\right) K_t. \tag{55}$$

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Equating (53) and (55), the market clearing condition is

$$(\mu - \frac{\rho}{x} + \pi)(1 - x) + \tau = g_M \frac{1 - x}{x} + \frac{\tau}{x}. \tag{56}$$

By (49), $\mu + \pi = x\sigma^2$. So (56) writes

$$(x\sigma^2 - \frac{\rho}{x})(1 - x) = g_M \frac{1 - x}{x} + \tau \frac{1 - x}{x}.$$

Simplifying, this yields the rate of growth of money supply which must prevail in equilibrium when the government follows policy (τ, π).

$$g_M = \sigma^2 x^2 - \rho - \tau. \tag{57}$$

5.1.3   Equilibrium growth rate and principal's consumption

By definition, the growth rate of money is $g_M = g + \pi$. Equating this to (57) we obtain the equilibrium growth rate obtaining for policy (τ, π).

$$g = \sigma^2 x^2 - \rho - \tau - \pi. \tag{58}$$

By (55), the principal's consumption share of capital is

$$\gamma^P = g_M \frac{1 - x}{x} + \frac{\tau}{x}.$$

Substituting (57), we have

$$\gamma^P = \left(\sigma^2 x^2 - \rho\right) \frac{1 - x}{x} + \tau. \tag{59}$$

Summarizing the results derived above, we obtain the next proposition:

Proposition 13 *When the principal targets a constant inflation rate π and announces a constant tax rate τ, the only rational expectations equilibrium where prices are finite is as follows:*

- *Each agent consumes a constant fraction ρ of her wealth.*

- *Each agent holds a constant fraction $x = \frac{\mu + \pi}{\sigma^2}$ of her wealth in the risky asset and the complementary fraction in money.*

- *The growth rate of money supply is $g_M = \sigma^2 x^2 - \rho - \tau$.*

- *The growth rate of output is $g = \sigma^2 x^2 - \rho - \tau - \pi$, and*

- *The principal's consumption share is $\gamma^P = \left(\sigma^2 x^2 - \rho\right) \frac{1-x}{x} + \tau$.*

The proposition clarifies that for any couple of policy variables π and τ, there is a unique stationary equilibrium allocation. Moreover, this allocation is associated with the variables x, g_M, g, and γ^P characterized in the proposition.[17] However, we show below that only a subset of equilibrium allocations are information constrained optimal.

5.2   Implementation

To implement a second best allocation we need to find τ and π such that i) the dynamics of $u(e_t)$ in equilibrium is equal to that of ω_t in that second best allocation and ii) the consumption of the principal in equilibrium is equal to the consumption of the principal in that second best allocation. Let us look first at the identification of the utility of the agent in the second best and in equilibrium. Proposition 12 implies that in the second best the dynamics of an agent's utility is

$$d\omega_t = \left(\frac{g}{\rho} - \frac{\rho \sigma^2 y^2}{2}\right) dt + \sigma y dB_t, \tag{60}$$

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[17] Instead of defining the principal's policy in terms of monetary (π) and fiscal (τ) policies, one could have equivalently defined it in terms of monetary (π) and budget (γ^P) policy. Because of the principal's budget constraint, stating the principal's consumption must be equal to the sum of seigneurage and tax revenues, setting τ is equivalent to setting γ^P

where

$$g = \mu - \rho - \frac{\rho \sigma^2 y}{1 + \rho \sigma^2 y^2}. \tag{61}$$

Turning to the equilibrium, by Ito's Lemma the dynamics of an agent's utility is

$$du(e_t) = u'(e_t)de_t + \frac{1}{2}u''(e_t)(de_t)^2.$$

By (52), this can be written as

$$du(e_t) = \frac{1}{\rho}\left[x\mu - (\rho + \tau + \pi(1-x))\right]dt - \frac{\sigma^2 x^2}{2\rho}dt + \frac{\sigma x}{\rho}dB_t. \tag{62}$$

For the equilibrium to implement the second best, we need to identify (60) and (62). For the Brownian term to be the same in the two equations, we need

$$x = \rho y. \tag{63}$$

Substituting the value of x from (49) into (63) the equality becomes

$$\frac{\mu + \pi}{\sigma^2} = \rho y.$$

So, to ensure that the equilibrium implements the second best allocation parametrized by y, the principal must target inflation rate π such that

$$\pi = \sigma^2 \rho y - \mu. \tag{64}$$

Once the two Brownian terms are equal, we need to identify the drifts, i.e., we must have

$$\frac{g}{\rho} - \frac{\rho \sigma^2 y^2}{2} = \frac{1}{\rho}\left[x\mu - (\rho + \tau + \pi(1-x))\right] - \frac{\sigma^2 x^2}{2\rho}. \tag{65}$$

After a few manipulations, explicited in the proof in the appendix, this is equivalent to

$$\tau = \sigma^2 \rho^2 y^2 \left(1 - \frac{\sigma^2 y}{1 + \rho \sigma^2 y^2}\right), \tag{66}$$

which, as shown in the proof in the appendix, also implies that the consumption of the principal is the same in equilibrium and in the second best. So we can state our next proposition:

Proposition 14 *Consider a second best allocation parametrized by y. This allocation can be decentralized as the competitive equilibrium associated with public policy (π, τ), where*

$$\pi = \sigma^2 \rho y - \mu, \tag{67}$$

and

$$\tau = \sigma^2 \rho^2 y^2 \left(1 - \frac{\sigma^2 y}{1 + \rho \sigma^2 y^2}\right)$$

As noted after Proposition 10, the optimal allocation involves $y > 0$. By (66) this implies that a laissez-faire policy with $\tau = 0$ cannot implement a second best allocation. That is non zero taxes or subsidies are necessary to implement second best optimal allocations.

Proposition 14 is a form of second welfare theorem: any Pareto optimal allocation can be decentralized as the competitive equilibrium for a particulary policy mix (π, τ). But there are major differences between Proposition 14 and the classical second welfare theorem: First, in the classical welfare theorem, markets are perfect and complete. In contrast, in our analysis there are asymmetric information frictions, implying that markets are endogenously incomplete. Second, the classical second welfare theorem considers lumpsum taxes, which don't distort agents' behaviour. In contrast, in our analysis taxes are linear in wealth, and in conjunction with inflation, optimally affect agents' behaviour.

A key step to obtain Proposition 14 is equation (63) which states that $x = \rho y$. A priori, y and x are conceptually different objects. The former is the exposure of agents to their idiosyncratic risk in the optimal mechanism. The latter is the structure of agents' portfolio in market equilibrium, which is affected by π since inflation determines the relative

attractiveness of the safe asset. To implement the optimal mechanism, the inflation target π must be set such that (63) holds, because this ensures that agents have the same risk exposure in equilibrium and in the optimal mechanism.

When g_M is negative (monetary contraction), the principal uses taxes to finance his consumption and to "pump out" money from agents. This case is similar to our two-date model in that the value of money equals the sum of future taxes minus future public expenditures (primary surpluses).[18] However, for different parameter values, g_M can also be positive (monetary expansion), in which case the money stock grows without limit, and the primary surplus is negative. Then money can be viewed as a bubble: its value is positive, even though taxes are insufficient to cover the consumption of the principal. It is even possible that τ be negative, implying that the principal subsidizes the agents by distributing them part of the money he issues (helicopter money). This is sustainable when growth rate is sufficiently high.

6   Conclusion

This paper analyzes capital allocation and risk sharing between a principal and many agents. We assume that agents privately observe their individual output and can secretly consume some of it, as in Bolton and Scharfstein [9]. To provide agents with incentives to reveal truthfully their output, the optimal dynamic mechanism allocates more capital and more consumption to agents with better performance. Thus, while there is no aggregate risk, incentive compatibility precludes perfect insurance. Assuming logarithmic utility enables us to fully characterize the optimal dynamics of capital and consumption as well as the distribution of continuation utilities across agents.

Moreover, we show that the optimal dynamic mechanism can be implemented by market equilibrium with appropriate inflation target and tax rate. Inflation determines the attractiveness of the safe asset relative to the risky asset, and thus agents' holdings of the latter. Targeting an appropriate inflation rate gives agents the same risk exposure in equilibrium as in the optimal mechanism, so that the former implements the latter.

This implementation result is a form of second welfare theorem: For any Pareto optimal allocation, there exists a fiscal and monetary policy implementing that allocation in equilibrium. However, while in the classical welfare theorem, markets are perfect and complete, in our analysis markets are endogenously incomplete because of information asymmetry. Moreover, while in the classical second welfare theorem taxes are lumpsum so that they don't distort agents' behaviour, in our analysis taxes depend on wealth and optimally affect agents' behaviour. Finally note that we don't obtain a first theorem of welfare. Only a subset of the equilibria arising in our setting are information constrained Pareto optimum. In particular, the laissez-faire equilibrium, obtaining with no taxation and no public expenditure, is not information constrained Pareto optimal.

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[18]There is no discounting as the interest rate on money is 0.

Appendix A: Proofs

Proof of Proposition 1: The Lagrangian is

$$E\left[U(C_1^s) + U(C_2^s)\right] + \lambda\left(1 - E\left[C_1^s + C_2^s\right]\right),$$

where λ is the multiplier of the resource constraint. The first order condition with respect to C_t^s is $U'(C_t^s) = \lambda, \forall s, t$. So consumption is constant across types s and periods. Binding the resource constraint this yields $C_1^s = \frac{1}{2}$.

    QED

Proof of Proposition 2: The Lagrangian is

$$E\left[U(C_1^s) + U(C_2^s)\right] + \lambda\left(1 - E\left[C_1^s + C_2^s\right]\right)$$
$$+ \nu\left[U(C_1^H) + U(C_2^H) - U(C_1^L + 2\sigma) - U(C_2^L)\right],$$

where λ is the multiplier of the resource constraint and ν the multiplier of the incentive constraint. The first order condition with respect to C_t^H is:

$$U'(C_t^H) = \frac{\lambda}{1+\nu}, \forall t. \tag{68}$$

So $C_1^H = C_2^H$. The first order condition with respect to C_1^L is:

$$U'(C_1^L) - \nu U'(C_1^L + 2\sigma) = \lambda. \tag{69}$$

The first order condition with respect to C_2^L is:

$$U'(C_2^L) = \frac{\lambda}{1-\nu}, \tag{70}$$

which, with (68), implies $C_2^L < C_2^H$. Now, (69) rewrites as

$$(1-\nu)U'(C_1^L) + \nu\left(U'(C_1^L) - U'(C_1^L + 2\sigma)\right) = \lambda.$$

That is

$$U'(C_1^L) = \frac{\lambda}{1-\nu} - \frac{\nu}{1-\nu}\left(U'(C_1^L) - U'(C_1^L + 2\sigma)\right),$$

which implies

$$U'(C_1^L) < \frac{\lambda}{1-\nu}.$$

Together with (70) this implies $C_2^L < C_1^L$.

    QED

Proof of Proposition 3 Denoting by λ the multiplier of the resource constraint and μ the one of the IC constraint, the first order conditions give $C_1^H = C_2^H = \frac{1+\mu}{\lambda}, C_2^L = \frac{1-\mu}{\lambda}$, and

$$\frac{1}{C_1^L} - \frac{\mu}{C_1^L + 2\sigma} = \lambda.$$

Since the IC constraint is binding we can write $C_1^L + 2\sigma = \frac{C_1^H C_2^H}{C_2^L} = \frac{(1+\mu)^2}{\lambda(1-\mu)}$. Similarly the resource constraint is binding, giving $C_1^L = 2 - \frac{3+\mu}{\lambda}$ By eliminating C_1^L between these two equations, we obtain $\frac{1}{\lambda} = \frac{(1+\sigma)(1-\mu)}{2}$ The expressions of $C_1^H = C_2^H, C_2^L$ are immediately deduced. Finally, the cubic equation in μ is obtained by plugging the expression of λ into the first order condition with respect to C_1^L.

    Now we turn to the proof of the 4 properties stated in the proposition:

1. $C_1^H - C_1^L = 2\sigma - (1+\sigma)\mu(1+\mu)$ Using the equation defining μ we can write $\mu(1+\mu) = \frac{1+3\mu}{1+\mu}\frac{\sigma}{1+\sigma}$. Since $\mu < 1$, this is smaller than $\frac{2\sigma}{1+\sigma}$. This establishes property 1.

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2. $C_2^H - C_2^L = (1+\sigma)(1-\mu)\mu > 0.$

3. $C_2^H + C_2^L = (1+\sigma)(1-\mu) < 1$ since $\mu > \frac{\sigma}{1+\sigma}$.

4. $C_2^H + C_1^H = (1+\sigma)(1-\mu^2) < 1 + \sigma.$

This ends the proof of the proposition.
QED

Proof of Proposition 4: The Lagrangian of the maximization problem faced by agent s is

$$U(C_1^s) + U(C_2^s) + \lambda^s \left[R^s + \frac{m_0}{p} - (C_1^s + k_1^s + \tau(e^s)) \right],$$

where $e^s \equiv R^s + \frac{m_0}{p} - C_1^s - C_2^s$. The first order condition with respect to time 1 consumption is

$$U'(C_1^s) = \lambda^s. \tag{71}$$

The first order condition with respect to investment is

$$U'(C_2^s) = \lambda^s \left[1 - \tau'(e^s) \right]. \tag{72}$$

Substituting (71) in (72) yields

$$\frac{U'(C_2^s)}{U'(C_1^s)} = \left[1 - \tau'(e^s) \right].$$

Since in the optimal mechanism $C_1^H = C_2^H$ and $C_1^L > C_2^L$, in the implementation we must have $\tau'(e^H) = 0$ and $\tau'(e^L) > 0$.

Binding the agent's goods budget constraint (1) and aggregating across agents yields

$$E\left[C_1^s + C_2^s + S^s\right] = E\left[R^s\right]. \tag{73}$$

Now the binding time 1 resource constraint faced by the planner is

$$E\left[C_1^s + C_2^s\right] = 1. \tag{74}$$

(73) and (74) imply

$$E\left[S^s\right] = 0,$$

which means that the goods market clears at time 1.
QED

Proof of Proposition 5: Denoting by β the Lagrange multiplier associated to the constraint on capital and λ^i the one associated to the promise keeping constraint for agent i, the Lagrangian writes, up to a constant:

$$L = \int_0^\infty e^{-\rho t} \log c_t^P dt + \int_0^1 \int_0^\infty \lambda^i e^{-\rho t} \log c_t^i dt di - \beta \int_0^\infty e^{-\mu t} \left[\int_0^1 c_t^i di + c_t^P \right] dt.$$

We can derive the first order conditions:

$$\frac{e^{-\rho t}}{c_t^P} = \beta e^{-\mu t},$$

with respect to c_t^P and

$$\frac{\lambda^i e^{-\rho t}}{c_t^i} = \beta e^{-\mu t},$$

with respect to c_t^i. This yields $c_t^P = \frac{exp(\mu-\rho)t}{\beta}$ and $c_t^i = \frac{\lambda^i exp(\mu-\rho)t}{\beta}$. Multiplying by ρ the promise keeping condition, we obtain

$$\rho\omega^i = \rho \int_0^\infty e^{-\rho t} \log c_t^i dt = \log(\frac{\lambda^i}{\beta}) + \frac{\mu-\rho}{\rho}.$$

Thus $\frac{\lambda^i}{\beta} = c_0^i = \gamma^A exp(\rho\omega^i)$. Now, we can multiply by ρ the constraint on capital, giving

$$\rho K = \gamma^P K + \gamma^A \int_0^1 exp(\rho\omega^i) di.$$

thus we can express γ^P as a function of the ratio of $\int_0^1 exp(\rho\omega^i) di$ and K, which we denote by z :

$$\gamma^P = \rho - \gamma^A \frac{\int_0^1 exp(\rho\omega^i) di}{K} = \rho - \gamma^A z.$$

Total consumption is thus $\rho K e^{(\mu-\rho)t}$. The dynamics of capital is:

$$\dot{K}_t = \mu K_t - \rho K e^{(\mu-\rho)t},$$

which gives after integration $K_t = Ke^{(\mu-\rho)t}$. The optimal allocation is thus stationary: individual consumptions and aggregate capital all grow at rate $\mu - \rho$. Similarly $\rho\frac{d\omega_t^i}{dt} = \mu - \rho$.
    QED

Proof of Proposition 7: Let $u = (c^A, c^P, y)$ be an admissible control, we will denote

$$J_\lambda^u = \int_0^\infty e^{-\rho t} \left(\log c_t^P + \lambda(K_t, \mathbb{P}_{\omega_t}) \left(K_t - \int y_t(\omega) c_t^A(w) d\mathbb{P}_{\omega_t}(\omega) \right) \right) dt,$$

and

$$J^u = \mathbb{E} \left[\int_0^\infty e^{-\rho t} \log(c_t^P) dt \right] = \int_0^\infty e^{-\rho t} \hat{c}_t^P dt$$

For every Lagrange multiplier λ, we have $V_\lambda = J_\lambda^{u_\lambda} \ge J_\lambda^u$. In particular, for $\lambda = \lambda_0$ and $u \in \mathcal{K}$, we have $J^{u_{\lambda_0}} = J_{\lambda_0}^{u_{\lambda_0}} \ge J_{\lambda_0}^u = J^u$ and since $u_{\lambda_0} \in \mathcal{K}$, the proof is complete.
    QED

Proof of Proposition 9: Fix $\mu \in \mathbb{P}_2(\mathbb{R})$ and a Lagrange multiplier λ and consider some arbitrary control $u(K_t, \mathbb{P}_{\omega_t}, \omega_t)$. We apply Itô's formula to $v^\lambda(K_t, \mathbb{P}_{\omega_t^\mu})$ between $s = 0$ and $s = t$ for $t > 0$.

$$e^{-\rho t} v^\lambda(K_t, \mathbb{P}_{\omega_t^\mu}) = v(K, \mu) + \int_0^t e^{-\rho s} \left(-\rho v^\lambda(K_s, \mathbb{P}_{\omega_s^\mu}) + v_K(K_s, \mathbb{P}_{\omega_s^\mu}) \left(\mu K - c^P - \int c^A(\omega) d\mathbb{P}_{\omega_s^\mu}(\omega) \right) \right) ds$$

$$+ \int_0^T e^{-\rho s} \int \partial_\omega \delta v^\lambda [K_s, \mathbb{P}_{\omega_s^\mu}](\omega)(\rho\omega - \log c^A(\omega)) \mathbb{P}_{\omega_s^\mu}(d\omega)$$

$$+ \int_0^t e^{-\rho s} \int \partial_{\omega\omega} \delta v^\lambda [(K_s, \mathbb{P}_{\omega_s^\mu}](\omega) \frac{\sigma^2}{2} y^2(\omega) \mathbb{P}_{\omega_s^\mu}(d\omega) ds.$$

We deduce from the Bellman equation (25) satisfied by v^λ that

$$v^\lambda(K, \mu) \ge e^{-\rho t} v^\lambda(K_t, \mathbb{P}_{\omega_t^\mu}) + \int_0^t e^{-\rho s} \left(\log(c_{\lambda,s}^P) + \lambda(K_s, \mathbb{P}_{\omega_s^\mu}) \left(K_s - \int y_{\lambda,s}(\omega) c_{\lambda,s}^A(\omega) \mathbb{P}_{\omega_s^\mu}(d\omega) \right) \right) ds.$$

Letting t tend to $+\infty$, we obtain using the transversality condition

$$v^\lambda(K, \mu) \ge \int_0^\infty e^{-\rho s} \left(\log(c_{\lambda,s}^P) + \lambda(K_s, \mathbb{P}_{\omega_s^\mu}) \left(K_s - \int y_{\lambda,s}(\omega) c_{\lambda,s}^A(w) d\mathbb{P}_{\omega_s^\mu}(\omega) \right) \right) ds = J_\lambda^u.$$

Since the control is arbitrary, we obtain

$$v^\lambda(K, \mu) \ge V_\lambda.$$

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On the other hand, let us apply the same Itô's argument with the control u_λ^* attaining the maximum in (25), we obtain

$$v^\lambda(K, \mu) = J_\lambda^{u_\lambda^*} \le V_\lambda,$$

which yields that $v^\lambda = V_\lambda$. We conclude the proof by applying Proposition 7.
 QED

Proof of Proposition (10): To obtain the dynamics of Z_t, we substitute $\gamma^A = 1/(yz)$ in $C_t^A(\omega) = \gamma^A \exp(\rho\omega_t)$, and then substitute the resulting expression into (23), which yields

$$d\omega_t = \log(yz)dt + \sigma y\, dB_t. \tag{75}$$

(75) and $Z_t = \mathbb{E}[\exp(\rho\omega_t)]$ yield

$$Z_t = Z_0\mathbb{E}\left[\exp(\rho\left(\log(yz)t + \sigma yB_t)\right)\right] = Z_0 \exp\left(\left(\rho\log(yz) + \frac{\rho^2\sigma^2 y^2}{2}\right)t\right), \tag{76}$$

which gives

$$\frac{dZ_t}{Z_t} = \left(\rho\log(yz) + \frac{\rho^2\sigma^2 y^2}{2}\right)dt. \tag{77}$$

By (28) and (77), equality of the growth rates of K_t and Z_t means that

$$\mu - \gamma^P - \frac{1}{y} = \rho\log(yz) + \frac{\rho^2\sigma^2 y^2}{2}. \tag{78}$$

The restricted principal's problem is thus characterized by the following maximization problem

$$V(K, \mathbb{P}) = \sup_{\gamma_P, y} \int_0^{+\infty} e^{-\rho t}\log(\gamma^P K_t)dt, \tag{79}$$

under the constraint (78) and the dynamics of capital

$$K_t = K exp((\mu - \gamma^P - \frac{1}{y})t). \tag{80}$$

Substituting K_t from (80) into (79), the latter writes

$$V(K, \mathbb{P}) = \sup_{\gamma_P, y} \int_0^{+\infty} [e^{-\rho t}\left(\log(\gamma_P K) + (\mu - \gamma^P - \frac{1}{y})t\right)]dt, \text{ s.t., (78)}. \tag{81}$$

Easy computations then show that (81) can be rewritten as

$$\rho V(K, \mathbb{P}) = \log K + \sup_{\gamma_P, y}\left(\log\gamma_P + \frac{\mu - \gamma^P - \frac{1}{y}}{\rho}\right), \text{ s.t., (78)}. \tag{82}$$

Using (78) we can express γ^P as a function of y and z

$$\gamma^P = \mu - \frac{1}{y} - \rho\left(\log(yz) + \frac{\rho\sigma^2 y^2}{2}\right).$$

Substituting the value of γ^P into (82), the latter writes as

$$\rho V(K, \mathbb{P}) = \log K + \sup_y\left(\log\left(\mu - \frac{1}{y} - \rho\left(\log(yz) + \frac{\rho\sigma^2 y^2}{2}\right)\right) + \log(yz) + \frac{\rho\sigma^2 y^2}{2}\right). \tag{83}$$

There exists a solution to (83) when the feasible set is non empty, i.e. when it is possible to find values of y for which the argument of the first log is positive. This is equivalent to

$$z < z_{max} := \max_y \frac{1}{y}exp[\frac{\mu}{\rho} - \frac{1}{\rho y} - \frac{\rho\sigma^2 y^2}{2}]. \tag{84}$$

Taking the first order condition in (83) and denoting

$$v(z) := \frac{1}{\rho}\sup_y\left(\log\left(\mu - \frac{1}{y} - \rho\left(\log(yz) + \frac{\rho\sigma^2 y^2}{2}\right)\right) + \log(yz) + \frac{\rho\sigma^2 y^2}{2}\right)., \tag{85}$$

we obtain that $\rho V(K, \mathbb{P}) = logK + \rho v(z)$. QED

Proof of Proposition 12: To prove Point 1 in Proposition 12 we start by observing that (22) states that the growth rate of capital is

$$g = \mu - \frac{\int c^A(\omega)d\mathbb{P}(\omega)}{K} - \frac{c^P}{K}$$

and that (26) states that

$$c^P = \gamma^P K, c^A(\omega) = \gamma^A \exp(\rho\omega).$$

Substituting the latter in the former, we have

$$g = \mu - \frac{\gamma^A \int \exp(\rho\omega)d\mathbb{P}(\omega)}{K} - \gamma^P.$$

By (27), this is

$$g = \mu - \gamma^A \frac{Z}{K} - \gamma^P. \tag{86}$$

As explained in the analysis of the restricted problem, (26) and (27) imply $\frac{Z_t}{K_t}$ is a constant, denoted by z, and $\gamma^A = \frac{1}{yz}$. Substituting in (86) yields

$$g = \mu - \frac{1}{y} - \gamma^P.$$

Substituting γ^P from (34), we obtain Point 1 in Proposition 12.

To prove Point 2 in Proposition 12, we start by recalling that (75) states

$$d\omega = \log(yz)dt + \sigma y dB_t$$

and that (78) implies

$$\log(yz) = \frac{\mu - \gamma^P - \frac{1}{y}}{\rho} - \frac{\rho\sigma^2 y^2}{2}.$$

Noting that the first term on the right-hand side is $\frac{g}{\rho}$, we obtain Point 2 in Proposition 12.

Point 3 in Proposition 12 is just a restatement of (34), while Points 4 and 5 are established at the beginning of the analysis of the restricted problem.

QED

Proof of Proposition 14: Replacing y by $\frac{x}{\rho}$ in the left-hand side, (65) becomes

$$\frac{g}{\rho} - \frac{\sigma^2 x^2}{2\rho} = \frac{1}{\rho}\left[x\mu - (\rho + \tau + \pi(1 - x))\right] - \frac{\sigma^2 x^2}{2\rho}.$$

The second terms on both sides are the same and cancel out, so we are left with

$$g = x\mu - (\rho + \tau + \pi(1 - x)). \tag{87}$$

Replacing y by $\frac{x}{\rho}$ in (61), the growth rate prevailing in the second best writes as

$$g = \mu - \rho - \frac{\sigma^2 x}{1 + \frac{\sigma^2}{\rho}x^2}. \tag{88}$$

Substituting $\pi = x\sigma^2 - \mu$ (from the definition of x), the right-hand side of (87) is

$$\mu - \rho - \tau - x\sigma^2(1 - x).$$

So, (87) writes

$$\mu - \rho - \frac{\sigma^2 x}{1 + \frac{\sigma^2}{\rho}x^2} = \mu - \rho - \tau - \sigma^2 x(1 - x),$$

which is equivalent to

$$\tau = \sigma^2 x^2 \left(\frac{1 - \frac{\sigma^2}{\rho}x(1 - x)}{1 + \frac{\sigma^2}{\rho}x^2}\right),$$

27

which yields (66).

Implementation also requires that the equilibrium sale of goods by agents, which is equal to the principal's consumption in equilibrium, be equal to the principal's consumption in the second best

$$\mathbb{E}[ds_t] = \gamma^P K_t dt, \tag{89}$$

where γ^P is given by (37) which substituting $y = x/\rho$ is

$$\gamma^P = \rho - \frac{1}{x/\rho + \sigma^2 x^3/\rho^2}. \tag{90}$$

By (53), (89) is equivalent to

$$(\mu - \frac{\rho}{x} + \pi)(1 - x) + \tau = \gamma^P,$$

that is

$$\tau = \gamma^P - (1 - x)(\sigma^2 x - \frac{\rho}{x}). \tag{91}$$

Substituting in (91) the value of γ^P from Proposition 12, this yields

$$\tau = \rho - \frac{1}{x/\rho + \sigma^2 x^3/\rho^2} - (1 - x)(\sigma^2 x - \frac{\rho}{x}),$$

which simplifies to

$$\tau = \sigma^2 x^2 \frac{1 - \frac{\sigma^2}{\rho} x(1 - x)}{1 + \sigma^2 x/\rho}.$$

which, because $x = \rho y$, is equivalent to (66).

QED

Appendix B: Differential calculus in the Wasserstein space

Consider a real-valued function F defined on $\mathcal{P}_2(\mathbb{R})$ the set of probability measures on $\mathbb{R}$ with finite second moment. We endow $\mathcal{P}_2(\mathbb{R})$ with the Wasserstein distance. For $\mu, \nu \in \mathcal{P}_2(\mathbb{R})$, $\Pi(\mu, \nu)$ is the set of transport plans, that is, probability measures on $\mathbb{R} \times \mathbb{R}$ with respective marginals μ and ν. The Wasserstein distance W_2 on $\mathcal{P}_2(\mathbb{R})$ is defined as the square root of

$$\min_{\gamma \in \Pi(\mu, \nu)} \int_{\mathbb{R}^2} |y - x|^2 \, d\gamma(x, y).$$

To apply a verification argument for the principal problem, we are interested in Itô's formula for F to describe the dynamic $t \to F(\mathbb{P}_{w_t})$, where $\mathbb{P}_{w_t}$ is the probability distribution of the process $(w_t)_t$ given by Equation (23). Itô's formula for F naturally requires differential calculus on the space of measures. We start by introducing the two notions of first variation and L-differentiability for functions of measures relying on the convexity of $\mathcal{P}_2(\mathbb{R})$ (for a more rigorous treatment, see for instance the standard references [12],[11] and [40]).

Definition 1 *We will say that*

- *A function F admits a first variation at $\mu \in \mathcal{P}_2(\mathbb{R})$ if there exists a real-valued and continuous function $\delta F[\mu] : \mathbb{R} \to \mathbb{R}$ such that for all ν in $\mathcal{P}_2(\mathbb{R})$, we have*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(F((1 - \varepsilon)\mu + \varepsilon\nu) - F(\mu) \right) = \int_{\mathbb{R}} \delta F[\mu](\omega) \, d(\nu - \mu)(\omega).$$

- *A function F is L-differentiable at $\mu \in \mathcal{P}_2(\mathbb{R})$ if the function $\delta F[\mu]$ is twice differentiable on $\mathbb{R}$ and we will denote $\partial_x \delta F[\mathbb{P}]$ and $\partial_{xx} \delta F[\mathbb{P}]$ its first and second L-derivatives.*

For a function F that is L-differentiable, Itô's formula associated to the dynamic $t \to F(\mathbb{P}_{w_t})$ takes the following form, see [12], Chapter 5, Th. 5.99,

$$F(\mathbb{P}_{w_t}) = F(\mathbb{P}_{w_0}) + \int_0^t \mathbb{E}\left[\partial_x \delta F[\mathbb{P}_{w_s}](w_s)(\rho w_s - \log c^A(K_s, \mathbb{P}_{w_s}, w_s)) \right] ds$$

$$+ \frac{1}{2} \int_0^t \mathbb{E}\left[\partial_{xx} \delta F[\mathbb{P}_{w_s}](w_s)\sigma^2 y^2(K_s, \mathbb{P}_{w_s}, w_s) \right] ds \tag{92}$$

A general class of L-differentiable functions can be described as follows. Let ϕ be a twice continuously differentiable function on $\mathbb{R}$ with quadratic growth and v a continuously differentiable function on $\mathbb{R}$. We consider the function F defined on $\mathcal{P}_2(\mathbb{R})$ by

$$F(\mu) = v\left(\int_{\mathbb{R}} \phi(x)\mu(dx)\right).$$

Then, F is L-differentiable with

$$\delta F[\mu] = v'\left(\int_{\mathbb{R}} \phi(x)\mu(dx)\right)\phi, \ \partial_x\delta F[\mu] = v'\left(\int_{\mathbb{R}} \phi(x)\mu(dx)\right)\phi' \text{ and } \partial_{xx}\delta F[\mu] = v'\left(\int_{\mathbb{R}} \phi(x)\mu(dx)\right)\phi''.$$

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