Monetary Economics: 3

Generation 3: Divisble Assets

Randall Wright

Generation 3: Divisible Assets

Replace with $m \in \{0, 1\}$ with $m \in \{0, 1, 2...\}$ or $m \in [0, \infty)$.
Issue: how do we handle the dist'n of liquidity $G(m)$?

- Computational work by Molico, Chiu, Wallace, Zhu etc.
- Theoretical devices by Shi, Menzio, Lagos-Wright etc.

Computation is great, but simple models are also useful for derving results and developing intuition.

Here we mainly study the LW alternating-market model.

- It is realistic, tractable, easy to integrate with standard macro or GE, and amenable to quantitative analysis;
- It captures the asynchronicity of expenditures and receipts key to any analysis of money or credit.

The Molico Model

$$V(m) = \alpha_1 \int \{u[q(m, \tilde{m})] + \beta V[m - d(m, \tilde{m})]\} dG(\tilde{m})$$
$$+ \alpha_0 \int \{-c[q(\tilde{m}, m)] + \beta V[m + d(\tilde{m}, m)]\} dG(\tilde{m})$$
$$+ (1 - \alpha_1 - \alpha_0)\beta V(m),$$

assuming stationarity, where $\alpha_1 = prob(\text{buy})$, $\alpha_0 = prob(\text{sell})$, and:

- $q(m, \tilde{m})$ and $d(m, \tilde{m})$ are quantity of goods and dollars if you buyer from seller with $\tilde{m}$;
- $q(\tilde{m}, m)$ and $d(\tilde{m}, m)$ are similar if you sell to buyer with $\tilde{m}$;

Note: we do not include barter or a return $\rho(m)$, but one could.

Simplification

$$
\begin{aligned}
V(m) = \alpha_1 \int \{u[q(m,\tilde{m})] + \beta V[m - d(m,\tilde{m})]\} \, dG(\tilde{m}) \\
+ \alpha_0 \int \{-c[q(\tilde{m},m)] + \beta V[m + d(\tilde{m},m)]\} \, dG(\tilde{m}) \\
+ (1 - \alpha_1 - \alpha_0)\beta V(m)
\end{aligned}
$$

$\Rightarrow$

$$
V(m) = \alpha_1 \int S_1(m,\tilde{m}) \, dG(\tilde{m}) + \alpha_0 \int S_0(\tilde{m},m) \, dG(\tilde{m}) + \beta V(m)
$$

where

$$
\begin{aligned}
S_1(m,\tilde{m}) &= u[q(m,\tilde{m})] + \beta V[m - d(m,\tilde{m})] - \beta V(m) \\
S_0(m,\tilde{m}) &= c[q(m,\tilde{m})] + \beta V[m + d(m,\tilde{m})] - \beta V(m)
\end{aligned}
$$

Terms of Trade in Molico Model

- In a meeting where buyer has m and seller has $\tilde{m}$, Nash bargaining implies

$$\max_{q,d} S_1 \left(m, \tilde{m}\right)^{\theta} S_0 \left(m, \tilde{m}\right)^{1-\theta}$$
$$\text{st } 0 \leq q \leq \bar{q} \text{ and } 0 \leq d \leq m$$

- Solution is $q = q\left(m, \tilde{m}\right)$ and $d = d\left(m, \tilde{m}\right)$.
- Other solution concepts can be used
- In fact Molico used take-it-or-leave-it offers by buyers.

Equilibrium in Molico Model

A stationary equil is a list of time-invariant functions

$$\langle V(m), q(m, \tilde{m}), d(m, \tilde{m}), G(m) \rangle$$

such that

1. $V(m)$ solves the DP eqn;
2. $q(m, \tilde{m})$ and $d(m, \tilde{m})$ solves the bargaining problem;
3. $G(m)$ is a stationarity distribution for the process

$$m' = \begin{cases} m - d(m, \tilde{m}) & \text{w/prob } \alpha_1 G'(\tilde{m}) \\ m + d(\tilde{m}, m) & \text{w/prob } \alpha_0 G'(\tilde{m}) \\ m & \text{w/prob } 1 - \alpha_0 - \alpha_1 \end{cases}$$

Applications

Molico solves the model on the computer and studies:

- price dispersion where $p(m, \tilde{m}) = d(m, \tilde{m}) / q(m, \tilde{m})$;
- monetary policy – lump sum transfers of m

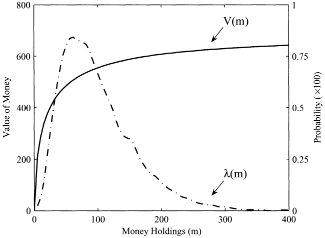


FIGURE 1

EQUILIBRIUM VALUE FUNCTION AND DISTRIBUTION OF MONEY HOLDINGS

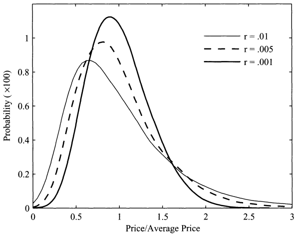
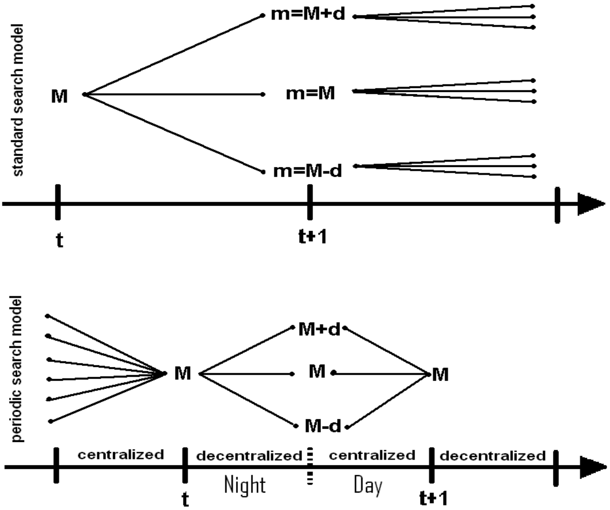


FIGURE 7

NORMALIZED DISTRIBUTION OF PRICES

Understanding the Distribution



The LW Alternating-Market Model

Each period $t = 0, 1, \ldots$ there are two subperiods and two markets:

- DM similar to earlier models, with value fn $V(m)$
- CM similar to standard GE, with value fn $W(m)$

In CM agents trade a numeraire good, labor and assets:

$$W(m) = \max_{x,\ell,\hat{m}} \left\{ \tilde{U}(x, 1-\ell) + \beta V(\hat{m}) \right\} \text{ st } x = \ell + \phi(m - \hat{m}) - T$$

Lemma: Suppose $\tilde{U} \in \mathcal{U}$. Then the the CM problem satisfies:

1. history independence, $\hat{m} \perp m$;
2. linearity, $W'(m) = \phi$.

If agents are homogeneous part 1 $\Rightarrow$ dist′n $G(m)$ is degenerate.

A Nice Class of Preferences

Wong (RES): The class $\mathcal{U}$ contains any $\tilde{U}$ with $\tilde{U}_{11}\tilde{U}_{22} = \tilde{U}_{12}^2$, including, e.g.,

1. Quasi-linear: $\tilde{U} = U(x) + 1 - \ell$ or $\tilde{U} = x + U(1-\ell)$
2. Cobb-Douglas: $\tilde{U} = x^a(1-\ell)^{1-a}$
3. CES: $\tilde{U} = [x^a + (1-\ell)^a]^{1/a}$

The results also hold for any increasing, concave $\tilde{U}$ in models with indivisible labor, $\ell \in \{0,1\}$; more on this below

Caveat: In any case, Lemma assumes interior solutions, *at least in some periods*.

Proof of Lemma: For simplicity consider $\tilde{U} = U(x) - \ell$ so that

$$W(m) = \max_{x,\ell,\hat{m}} \{U(x) - \ell + \beta V_+(\hat{m})\} \text{ st } x = \ell + \phi(m - \hat{m}) - T$$

Eliminate ℓ using budget equation to get

$$W(m) = \phi m - T + \max_x \{U(x) - x\} + \max_{\hat{m}} \{-\phi \hat{m} + \beta V_+(\hat{m})\}$$

Hence $W'(m) = \phi$ and FOC $\phi = \beta V'_+(\hat{m})$ implies $\hat{m} \perp m$. ■

Alternative Proof: FOC from the general case imply

$$\frac{\partial \hat{m}}{\partial m} = A\left(\tilde{U}_{11}\tilde{U}_{22} - \tilde{U}_{12}^2\right),$$

where $A = ...$ ■

DM Terms of Trade

Consider Nash bargaining

$$\max \left[u\left(q\right)-\phi d\right]^{\theta}\left[-c\left(q\right)+\phi d\right]^{1-\theta} \text{ st } d \leq m,$$

This is simple because, by above Lemma, the surpluses are

$$\begin{aligned} S_1 &= u\left(q\right)+W\left(m-d\right)-W\left(m\right)=u\left(q\right)-\phi d \\ S_0 &= W\left(m+d\right)-W\left(m\right)-c\left(q\right)=\phi d-c\left(q\right). \end{aligned}$$

Note: different from Molico model, sol'n $\left(q,d\right)$ does not depend on seller's $\tilde{m}$, and depends on buyer's m iff $d \leq m$ binds.

Lemma: $d \leq m$ binds on the equilibrium path (but for one exceptional case)

DM Value

Given $d = m$, the Nash bargaining solution is $z = v(q)$ where $z \equiv \phi m$ denotes real money and

$$v(q) = \frac{\theta u'(q)\, c(q) + (1-\theta)\, c'(q)\, u(q)}{\theta u'(q) + (1-\theta)\, c'(q)}$$

Note this looks like the model with indivisible money with z replacing Δ

We can use various other mechanisms, e.g., Kalai bargaining is

$$v(q) = \theta c(q) + (1-\theta) u(q).$$

Or just leave $v(q)$ as any function with $v(0) = 0$ and $v'(q) > 0$.

Monetary Mechanisms

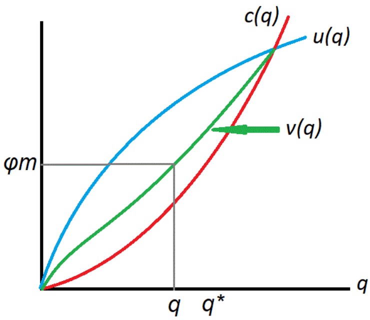
Let q^* solve $u'(q^*) = c'(q^*)$ and define d^* by $\phi d^* = v(q^*)$.
Then impose $c(q) \leq v(q) \leq u(q)$ plus:

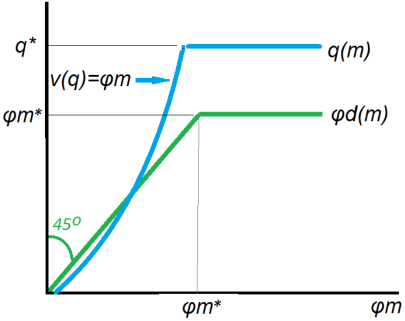
- if $m \geq d^*$ then $q = q^*$ and $d = d^*$.
- if $m < d^*$ then $d = m$ and q solves $v(q) = \phi m$.

Intuition:

- If you can afford q^* take it, and payment $p^* = v(q^*)$ is determined by v.
- Otherwise pay what you can $d = m$, and $q = v^{-1}(\phi m)$ is determined by v

These properties of $v(q)$ hold for Nash, Kalai, Walras, HKW ....
and can also be derived from simple axioms.





Marginal Value of Money

For any bargaining sol'n $\phi m = v(q)$,

$$V(m) = W(m) + \alpha_1 \{u[q(m)] - v[q(m)]\} + \alpha_0 \{-c[q(M)] - v[q(M)]\}.$$

Since the last term depends on $\tilde{m} = M$ but not m,

$$V'(m) = W'(\hat{m}) + \alpha_1 \left[u'(q) - v'(q)\right] q'(m).$$

Now plug in $W'(m) = \phi$ and, from bargaining sol'n,

$$q'(m) = \begin{cases} \phi / v'(q) & \text{if } q < q^* \\ 0 & \text{if } q \geq q^* \end{cases}$$

Demand for Money

$$V'(m) = \phi + \alpha_1 \left[u'(q) - v'(q) \right] \frac{\phi}{v'(q)} = \phi \left[1 + \alpha_1 \lambda(q) \right]$$

where $\lambda(q) =$ *liquidity premium* $=$ Lagrange multiplier on $d \leq m$,

$$\lambda(q) \equiv \begin{cases} u'(q)/v'(q) - 1 & \text{if } q < q^* \\ 0 & \text{if } q \geq q^* \end{cases}$$

Update one period and plug in $\phi = V'_+(m) \Rightarrow$ Euler eqn

$$\phi_t = \beta \phi_{t+1} \left[1 + \alpha_1 \lambda(q_{t+1}) \right].$$

Given path for ϕ_t this $\Rightarrow$ demand for q_t and $m_t = v(q_t)/\phi_t$.

Asset-Pricing Equation

Use $v(q) = \phi m$, let $L(\phi m) = \lambda \circ v^{-1}(\phi m)$ and set $m = M$ to get

$$\phi_t = \beta \phi_{t+1} \left[1 + \alpha_1 L(\phi_{t+1} M_{t+1}) \right].$$

Given path for M this is a DE in asset price ϕ.

Or, assume $M_+ = (1 + \mu) M$ to get

$$z_t = \frac{\beta z_{t+1}}{1 + \mu_t} \left[1 + \alpha_1 L \left(z_{t+1} \right) \right].$$

Given path for μ this is a DE in liquidity (real balances) z.

Equilibrium

Given a path for M equil is a nonnegative, bdd path for ϕ, or equivalently z, solving DE.

From ϕ or z, get q from BS, x from $U'(x) = 1$ and

$$\ell = x - \phi m + \phi M_{+} + T$$

Given gov't budget eqn $G = T + \mu\phi M$ implies

$$\ell = x + G + \phi(m - M)$$

Obviously in the aggregate $\bar{\ell} = x + G$.

Money can be injected by $G > T$ or withdrawn by $T > G$.

Stationary Monetary Equilibrium

$$z_t = \Phi(z_{t+1}) \equiv \frac{\beta z_{t+1}}{1+\mu_t}\left[1+\alpha_1 L(z_{t+1})\right].$$

If μ_t is constant, consider SME, or steady state, $z = \Phi(z)$.
z constant $\Rightarrow 1+\pi = \phi/\phi_+ = 1+\mu$ (quantity theory).
One sol'n is $z = 0$; other sol'ns satisfy

$$1 = \frac{\beta}{1+\mu}\left[1+\alpha_1 L(z)\right]$$

where we impose $\mu > \beta - 1$, for this to make sense, but also consider the limit $\mu \to \beta - 1$.

Interest rates

- Let $1 + r$ define a real interest rate by asking how much x agents would give in CM_{t+1} for a unit of x in CM_t.
- Let $1 + i$ define a nominal interest rate by asking how much m agents would give in CM_{t+1} for a unit of m in CM_t.

Answers: $1 + r = U'(x_t)/\beta U'(x_{t+1})$, and the Fisher equation

$$1 + i = (1 + \pi)(1 + r)$$

In SME $1 + r = 1/\beta$ – also true out of SME bc $U'(x_t) = 1$.

In SME $\pi = \mu$, so $\mu > \beta - 1 \Leftrightarrow i > 0$ (the ZLB) and $i \to 0$ is the Friedman rule

Interpretation

SME condition can now be rewritten

$$i = \alpha_1 L(z)$$

which *simply* equates the MC and MB of money, but don't forget:

- i is the opportunity cost of foregone lending
- $\alpha_1 = \alpha\sigma$ incorporates search and matching probabilities
- $L = \lambda \circ v^{-1}$ is a multiplier that depends on u and v
- v incorporates the mechanism – eg, Nash, Kalai, Walras ...
- and behind it all is imperfect commitment/information.

Existence and Uniqueness

Prop: SME, $z > 0$ solving $i = \alpha_1 L(z)$, exists iff $i < \hat{\imath}$.

- Sometimes (eg Nash bargaining) $\hat{\imath} = \infty$
- Sometimes (eg Kalai bargaining) $\hat{\imath} < \infty$.

Clearly $L'(z) < 0 \Rightarrow$ uniqueness.

- Sometimes (eg Kalai bargaining) $L'(z) < 0$
- Sometimes (eg Nash bargaining) we're not sure

Prop: Even without $L'(z) < 0$, generically SME is unique and $\partial z / \partial i < 0$.

Simple Example

Kalai sol'n: $v(q) = \theta c(q) + (1-\theta) u(q) \Rightarrow$

$$\lambda(q) = \frac{u'(q)}{v'(q)} - 1 = \frac{\theta[u'(q) - c'(q)]}{\theta c'(c) + (1-\theta) u'(q)}$$

Easiest case: $\theta = 1$ (Kalai same as Nash) $\Rightarrow v(q) = c(q)$

$c(q) = q \Rightarrow \lambda(q) = u'(q) - 1$, $L(z) = u'(z) - 1$, MSS solves

$$i = \alpha_1 u'(q) \text{ or } i = a_1 u'(z).$$

MSS exists uniquely $\forall i > 0$, with $\partial q/\partial i < 0$, $q \to q^*$ as $i \to 0$.

General Kalai

$$i = \alpha_1 \lambda(q) = \frac{\alpha_1 \theta [u'(q) - c'(q)]}{\theta c'(c) + (1-\theta) u'(q)}$$

Easy to check $\lambda'(q) \simeq c'(q) u''(q) - u'(q) c''(q) < 0$.
MSS exists uniquely iff $\lim_{q \to 0} \lambda(q) > i/\alpha_1$ iff

$$i < \frac{\alpha_1 \theta [1 - \frac{c'(q)}{u'(q)}]}{\theta \frac{c'(q)}{u'(q)} + 1 - \theta} = \frac{\alpha_1 \theta}{1-\theta} \equiv \hat{\imath}.$$

Also $\partial q / \partial i = 1/\alpha_1 \lambda'(q) < 0$ and $q \to q^*$ as $i \to 0$.

Nash

$$v(q) = \frac{\theta u'(q)\, c(q) + (1-\theta)\, c'(q)\, u(q)}{\theta u'(q) + (1-\theta)\, c'(q)}$$

Implies

$$v'(q) = \frac{\theta(1-\theta)(u-c)(u'c'' - c'u'') + u'c'\left[\theta u' + (1-\theta)\, c'\right]}{\left[\theta u' + (1-\theta)\, c'\right]^2}$$

Now $\forall\theta > 0$ existence follows from $\lambda(q) \to \infty$ as $q \to 0$.

But λ' depends on u''' and c''' so uniqueness is trickier.

One can show $\partial q/\partial i < 0$ and $q \to q^*$ as $i \to 0$ iff $\theta = 1$.

Reformulating the Problem

$$\begin{aligned} W(z) &= \max_{x,\ell,\hat{z}} \{U(x) - \ell + \beta V_+(\hat{z})\} \text{ st } x = \ell + z - (1+\pi)\hat{z} - T \\ &= A_1 + \max_{\hat{z}} \{-(1+\pi)\hat{z} + \beta V_+(\hat{z})\} \end{aligned}$$

Use $V_+(\hat{z}) = A_2 + \alpha_1 \upsilon(\hat{z}) + (1-\alpha_1)\hat{z}$ where $\upsilon(\hat{z}) = u \circ v^{-1}(z)$,

$$\begin{aligned} W(z) &= A_3 + \max_{\hat{z}} \{-(1+\pi)\hat{z} + \beta\alpha_1 \upsilon(\hat{z}) + \beta(1-\alpha_1)\hat{z}\} \\ &= A_3 + \beta \max_{\hat{z}} \{-(1+\pi)(1+r)\hat{z} + \alpha_1 \upsilon(\hat{z}) + (1-\alpha_1)\hat{z}\} \\ &= A_3 + \beta \max_{\hat{z}} \{-(\alpha_1 + i)\hat{z} + \alpha_1 \upsilon(\hat{z})\}. \end{aligned}$$

This *decision problem* implies same z as SME.

Can it have multiple sol'ns? Not generically – if there are, say, 2 soln's, just tweak i and get back to 1 sol'n.

$\Rightarrow \exists! z > 0$ with $i = \alpha L(z)$ for generic i, with $\partial z / \partial i < 0$ i, except it may jump.

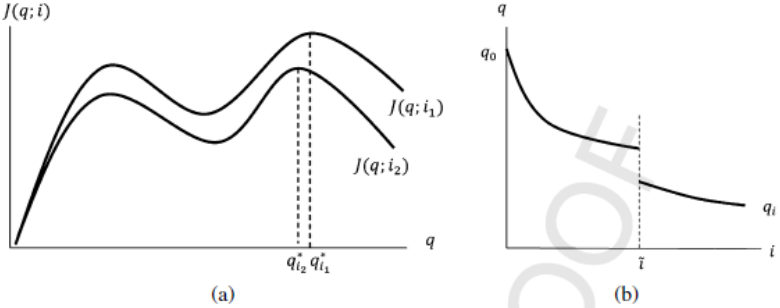


Fig. 1. (a) The objective function. (b) The solution q_i vs i.

Policy

- In SME higher $i \Rightarrow$ lower z and lower q.
- By Fisher eqn i moves one-for-one with π in SME.
- By def'n of SME π moves one-for-one with μ.

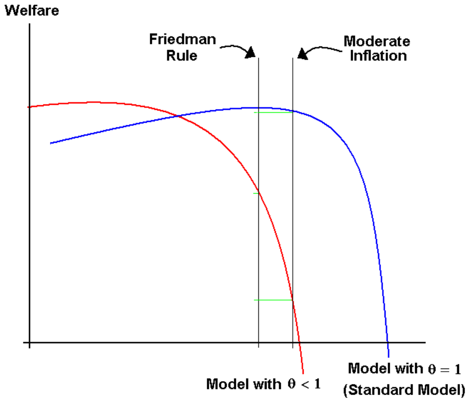
Hence it is equivalent in SME to target μ or π or i.

Standard mechanisms (Nash, Kalai, Walras...) $\Rightarrow q < q^* \ \forall i > 0$.

Since $\partial q / \partial i < 0$, optimal monetary policy is Friedman rule, $i = 0$, but that may or may not achieve q^*:

- $q = q^*$ at $i = 0$ with Kalai bargaining $\forall \ \theta > 0$
- $q = q^*$ at $i = 0$ with Nash bargaining iff $\theta = 1$.

This implies the ZLB problem can be big and the cost of inflation can be really big.



Quantifying the Effect

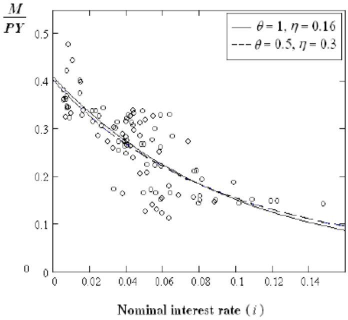
Consider $U(x) = \log(x)$, $c(q) = q$ and $u(q)$ like CRRA utility except $u(0) = 0$:

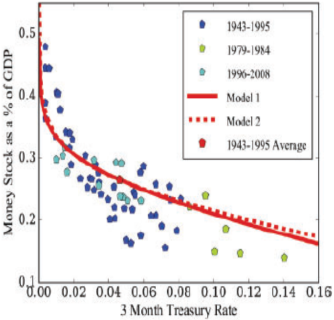
$$u(q) = A\frac{(q+b)^{1-a} - b^{1-a}}{1-a}$$

Set $\beta = 1/(1+r)$ to match (some) real interest rate, (a, A) to match money demand, and θ to match retail markup.
Set $\alpha\sigma$ arbitrarily but check how it matters.
Cost of $\pi = 10\%$ instead of Friedman rule: 5.0% of cons.
Standard finding CIA or MUF models: 0.5% of cons.

Microfoundations matter!

Empirical Money Demand





Interest on Money

Andolfatto (JET) pays interest $\tilde{\imath}$ on money, so

$$W(m) = \max\{...\} \text{ st } x = \ell + \phi m (1 + \tilde{\imath}) - \phi \hat{m} - T.$$

EC is $W'(m) = \phi(1 + \tilde{\imath})$, BS is $v(q) = \phi m (1 + \tilde{\imath})$ and EE is

$$\begin{aligned} \phi_{-1} &= \beta\phi(1 + \tilde{\imath})[1 + \alpha\lambda(q)] \\ (1 + \mu) M_{-1}\phi_{-1} &= \beta M \phi (1 + \tilde{\imath})[1 + \alpha\lambda(q)] \end{aligned}$$

Only $(1 + \tilde{\imath}) / (1 + \mu)$ matters for the equil set, and, in particular, in SME

$$\frac{1 + \mu}{1 + \tilde{\imath}} = \beta[1 + \alpha\lambda(q)]$$

Generalized Friedman Rule

Any $(\tilde{\imath}, \mu)$ with same $(1+\mu) / (1+\tilde{\imath}) \Rightarrow$ same q but different $z = v(q) / (1+\tilde{\imath})$.

Note $\lambda(q) = 0$ at $(\tilde{\imath}, \mu) = (0, \beta - 1)$ and $(1/\beta - 1, 0)$.

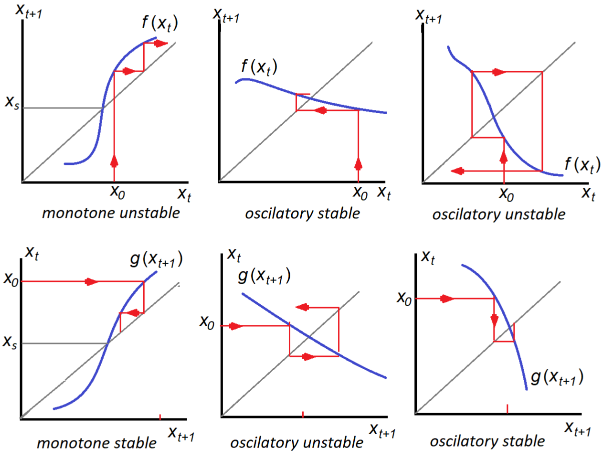
Any $(\tilde{\imath}, \mu)$ that gives the same q also implies the same

$$T = G + (\tilde{\imath} - \mu) \phi M = G + \frac{i - \mu}{1 + i} v(q).$$

- $\forall \mu$ $\lambda(q) = 0$ iff $\tilde{\imath} = i$ (same rate on liquid and illiquid assets)
- $\tilde{\imath} = \mu$ (pay for interest on M by printing M) is super neutral

Dynamics

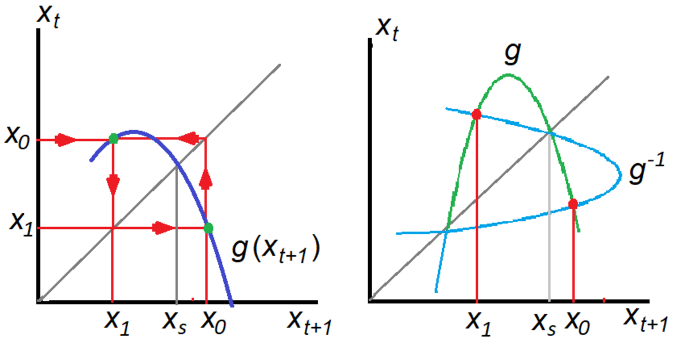
Theory $\Rightarrow$ dynamic system, $x_{t+1} = f(x_t)$ or $x_t = g(x_{t+1})$, with a few possibilities shown (see Azariadis 1993).



Dynamics: Cycles

Left: A 2-cycle, $x_1 = g(x_0)$ and $x_0 = g(x_1)$, i.e., a sol'n $x \neq x_s$ to $x = g^2(x)$, $g^2(x) \equiv g \circ g(x)$

Right: Clearly $g'(x_s) < -1 \not\Rightarrow$ there is a 2-cycle around x_s



Dynamics: 2-Cycles

Left: g^2 has three fixed points, x_s and the points of a 2-cycle
Right: 2-cycle is where g and g^{-1} intersect off the 45^o line

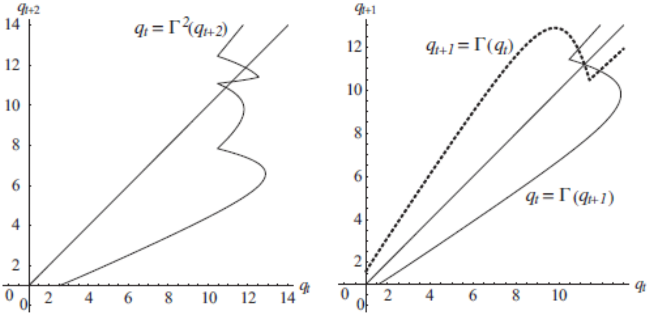


Fig. 7. Two-period cycles.

Dynamics: 3-Cycles

Here g^3 has seven fixed points, x_s and a pair of 3-cycle

Theorem: if there is a 3-cycle there are n-cycles $\forall n$, including $n = \infty$ (chaos)

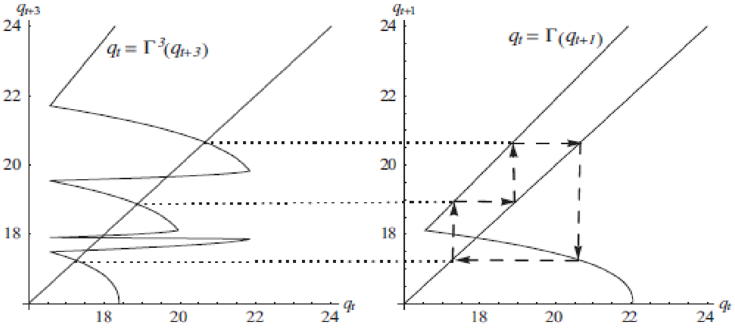


Fig. 9. Three-period cycles.

Chaotic Dynamics in 1st–Order System

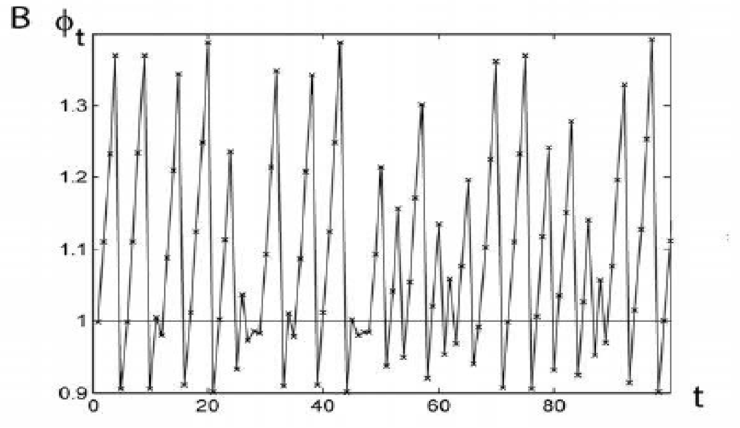
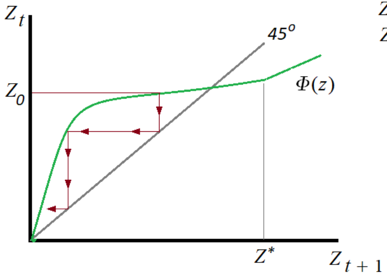


FIG. 4.—*A*, A three-period cycle; *B*, chaotic dynamics

Dynamics: Alternating-Market Asset-Pricing Model

There exist ME with $z_t \longrightarrow 0$, and sometimes with z_t cycling. Also notice the kink in $\Phi(z)$ at $z^* = v(q^*)$.



Alternative Specification

In LW, as in 1st and 2nd generation models, agents are buyers or sellers in DM depending on who they meet.

In RW, some are *always* buyers and others *always* sellers in DM.

- That cannot work in models with no CM because...

CM: same as before except sellers choose $\hat{m} = 0$

DM: arrival rates $\alpha_0 = \alpha(n)$ for sellers and $\alpha_1 = \alpha(n)/n$ for buyers where $n = n_1/n_0$ is tightness.

SME condition:

$$i = \frac{\alpha(n)}{n} L(z)$$

Now we can endogenize n with entry by one side of the market, say $k_s = V^s$.

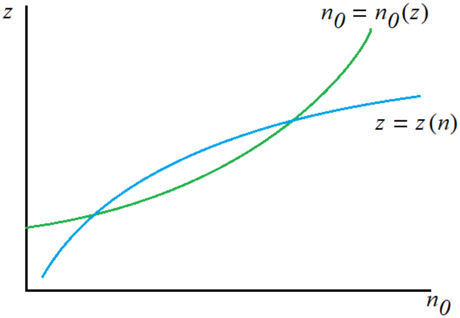


Figure: Multiple equil with entry by sellers in RW model

Real Assets

Geromichalos et al. replace fiat m with real asset a in fixed supply A, with price ψ and dividend ρ.

- Standard assets $\rho > 0$; toxic assets $\rho < 0$; fiat money $\rho = 0$.

$$W(a) = \max_{x,\ell,\hat{a}} \{U(x) - \ell + \beta V(\hat{a})\} \text{ st } x = \ell + \rho a + \psi(a - \hat{a}) - T$$

We still get $\hat{a} \perp a$, and linearity with $W'(m) = \rho + \psi$.
BS $v(q) = (\rho + \psi) p$, but $p \leq a$ *may or may not* bind:

$$\begin{aligned} a &\geq p^* \Rightarrow p = p^* \text{ and } q = q^* \\ a &< p^* \Rightarrow p = a \text{ and } q = v^{-1}(a) < q^* \end{aligned}$$

Results with Real Assets

Asset-pricing eqn changes as follows:

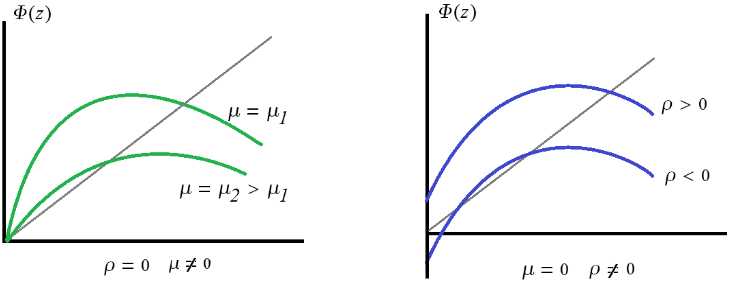
$$\begin{aligned}\phi_t &= \beta\phi_{t+1}\left[1+\alpha\sigma L\left(\phi_{t+1}M_{t+1}\right)\right]\\ \psi_t &= \beta\left(\rho+\psi_{t+1}\right)\left[1+\alpha\sigma L\left(\rho A+\psi_{t+1}A\right)\right]\end{aligned}$$

Now $z_t = \left(\rho+\psi_t\right)A$ gives total liquidity, and dynamic system changes as follows:

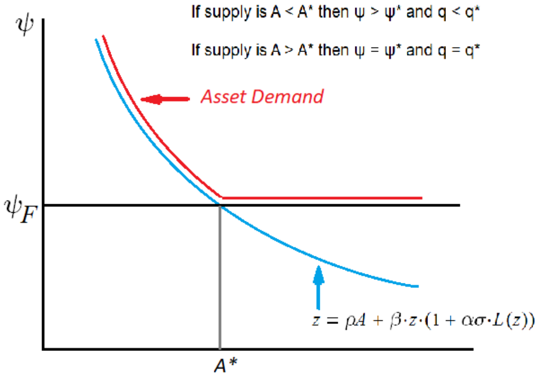
$$\begin{aligned}z_t &= \Phi\left(z_{t+1}\right) = \beta\frac{z_{t+1}}{1+\mu}\left[1+\alpha\sigma L\left(z_{t+1}\right)\right]\\ z_t &= \Phi\left(z_{t+1}\right) = \rho A+\beta z_{t+1}\left[1+\alpha\sigma L\left(z_{t+1}\right)\right]\end{aligned}$$

Note: μ affects the slope and ρ affects the intercept of the system – important for steady state and dynamics.

Dynamic system: left shows fiat money, for two money growth rates; right shows real assets, both standard and toxic.



Equil asset price is bounded below by fundamental $\psi_F = \rho / r$.



Lagos Asset Pricing Papers

- In several papers Lagos uses a variant with three goods:
  - q traded in DM; y and c traded in CM.
- Utility of q is $u(q)$, utility of y is y; utility of c is $U(c)$.
- q produced 1-for-1 with DM labor; y produced 1-for-1 with CM labor; c is *stochastic* dividend ρ on fixed-supply asset a.
- Labor gives constant marginal disutility 1.
- Use y as numeriare, p and ϕ for prices of c and a.
- Compared to baseline model, c and y are different goods.

Lagos Asset Pricing Model

- In the CM

$$\begin{aligned} W(a) &= \max_{c,y,h,\hat{a}} \{U(c) + y - h + \beta \mathbb{E} V(\hat{a})\} \\ \text{st } pc &= h - y + (p\rho + \phi) a - \phi \hat{a}. \end{aligned}$$

$$\Rightarrow W'(a) = p\rho + \phi,\ p = U'(c) \text{ and } \phi = \beta \mathbb{E} V'(\hat{a})$$

- In the DM

$$V(a) = W(a) + \alpha [u(q) - (p\rho + \phi) d].$$

where $v(q) = d(p\rho + \phi)$ and $d \leq a$ binds for low but not high realizations of ρ.

$$V'(a) = (p\rho + \phi)[1 + \alpha\lambda(q)].$$

Lagos Asset Pricing Equation

$$\phi = \beta \mathbb{E}(p_+ \rho_+ + \phi_+)\left[1 + \alpha \lambda\left(q_+\right)\right]$$

- Equilibrium: $\hat{a} = A$, $c = \rho A$, $p = U'(\rho A)$, q solves bargaining and ϕ solves EE.
- If ρ iid, could say ϕ constant, but we don't need that.
- Actually, Lagos writes it in terms of c not y using $\psi = \phi / p$,

$$U'(A\rho)\,\psi = \beta \mathbb{E}\, U'\left(A\rho_+\right)\left(\rho_+ + \psi_+\right)\left[1 + \alpha \lambda\left(q_+\right)\right]$$

  which looks like std asset-pricing, with U' on BS.
- Either version differs from $\phi = \beta \mathbb{E}(\rho_+ + \phi_+)\left[1 + \alpha \lambda\left(q_+\right)\right]$.
- Reason: Here c is random; baseline had x constant with h adjusting so $U'(x) = 1$.

Another Asset Pricing Equation

Go back to baseline but with ρ, ω and maybe even B random:

$$W(a) = \max_{x,h,\hat{a}} \{U(x) - Bh + \mathbb{E}\beta V_+(\hat{a})\} \text{ st } x = \omega h + \rho a + \phi a - \phi \hat{a}$$

$$\Rightarrow W'(a) = (\rho + \phi) B/\omega, \; U'(x) = B/\omega, \phi B/\omega = \mathbb{E}\beta V'_+(\hat{a}).$$

In the DM, $v(q) = d(\rho + \phi) B/\omega$ and $d \leq a$, so

$$V(a) = W(a) + \alpha [u(q) - d(\rho + \phi) B/\omega]$$

Hence EE is

$$\phi(B/\omega) = \mathbb{E}(B_+/\omega_+)\left(\rho_+ + \phi_+\right)[1 + \alpha\lambda(q_+)]$$

Or, using $U'(x) = B/\omega$,

$$U'(x)\phi = \mathbb{E}U'(x_+)\left(\rho_+ + \phi_+\right)[1 + \alpha\lambda(q_+)]$$

Safe and Risky Assets Lagos Model

$$\begin{aligned} W(a,b,\rho) &= \max_{c,y,h,\hat{a},\hat{b}} \left\{ U(c) + y - h + \beta \mathbb{E} V(\hat{a},\hat{b}.\hat{\rho}) \right\} \\ \text{st } pc &= h - y + \phi_a a + p\rho a + b - T - \phi_a a - \phi_b b \end{aligned}$$

where a is a tree-like asset with random ρ and b is a safe real bond.

FC's are

$$p = U'(c), \ \phi_a = \beta \mathbb{E} V_1(\hat{a},\hat{b}.\hat{\rho}) \text{ and } \phi_b = \mathbb{E} V_2(\hat{a},\hat{b}.\hat{\rho})$$

EC's are

$$W_1(a,b,\rho) = \phi_a + p\rho \text{ and } W_2(a,b,\rho) = 1$$

DM with Safe and Risky Assets

$$V(a,b,\rho) = (a,b,\rho) + \alpha\left[u(q) - p\right],\; v(q) = p \leq \bar{p}$$

where $\bar{p} = \chi_a(p\rho + \phi_a)a + \phi_b b$. Assuming $p \leq \bar{p}$ binds,

$$\begin{aligned}\phi_a &= \beta\mathbb{E}(p\rho + \phi_a)\left[1 + \alpha\chi_a\lambda(q)\right] \\ \phi_b &= \beta\mathbb{E}\left[1 + \alpha\chi_a\lambda(q)\right].\end{aligned}$$

If a is safe then

$$r_a = \frac{\rho U'(c)}{\phi_a} = \frac{r + \alpha\chi_a\lambda(q)}{1 + \alpha\chi_a\lambda(q)} \text{ and } r_b = \frac{1}{\phi_b} - 1 = \frac{r + \alpha\chi_b\lambda(q)}{1 + \alpha\chi_b\lambda(q)},$$

and $\chi_a = \chi_b \Rightarrow r_a = r_b$. Things are more interesting when a is not safe.