Sequentially Mixed Search and Equilibrium Price Dispersion*

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This version: July 2020

Abstract

Many markets feature sequentially mixed search (SMS), which has directed search followed by noisy matching with multiple offers. I construct a simple model of SMS, establish existence of a unique equilibrium, and analyze the novel implications of the equilibrium on quantities and price dispersion. Moreover, I show that an increase in the meeting efficiency widens price dispersion. An extension endogenizing search effort shows that the equilibrium is constrained inefficient, where search effort is inefficiently high and can be strategic complements among visitors. Under a mild condition, policies that restore efficiency should lean against the wind to manage aggregate demand and supply.

JEL classification: D40; D60; D83.
Keywords: Sequentially mixed search; Price dispersion; Efficiency.

* Address: Shouyong Shi, 502 Kern Building, Pennsylvania State University, University Park, PA 16802, USA. A version of this paper has been presented at the Society for Economic Dynamics (2019) and the theoretical advances in search and matching (organized by NBER and Princeton, 2019). I thank Leena Rudanko for discussing the paper and the participants of the conferences for comments.

1. Introduction

A real estate agent often asks a house buyer what price range and neighborhood the buyer is searching for. The agent gives a list of houses on the market that fit into the buyer's description and the buyer can visit more than one house on the list before making an offer. Similarly, when searching for a job, a worker has a desired range of wages and restricts the applications to such wages. The worker may be interviewed for several jobs that fit into the desired range before choosing whether to accept one of them. In the market for goods and services, a consumer narrows down search to a specific range of prices and conditions and, after getting a number of quotes in the range, chooses whether to trade. An example is online booking of airline tickets or hotel rooms.

The common feature of all these markets is *sequentially mixed search* (SMS) – meeting with directed search followed by noisy matching. In the *meeting* stage, individuals choose a feature of the price distribution to restrict search, such as the maximum price, with rational expectations that the meeting rate will depend on the specified feature. For example, increasing the maximum house price to search for increases the probability of finding such a house in any given neighborhood. In this stage a searcher cannot target a particular offer because many offers can have the same specified feature. The noisy matching stage is as in Burdett and Judd (1983, henceforth BJ). A searcher randomly receives a number of meetings all having the specified feature and chooses which one to accept.

Given the ubiquitous nature of SMS, it is surprising that it has not received much attention in the literature. Most papers in the literature formulate search either as purely directed, which excludes noisy matching in the second stage, or as purely undirected.[1] When mixed search is the focus of a small number of papers, the mixing is simultaneous across markets rather than sequential in the same market, e.g., Lester (2011), Godoy and Moen (2013), and Delacroix and Shi (2013). In these papers, some markets are open for directed search while other markets for random search simultaneously, but no searcher uses

[1] The literature on undirected search goes back at least to Diamond (1971). For directed search models, well-known examples are Peters (1991), Montgomery (1991), Moen (1997), Julien et al. (2000), Burdett et al. (2001), and Shi (2001). See additional references at the end of this Introduction.

both modes of search in any market or receives multiple offers.

I construct a simple model of SMS to study the positive and normative implications.[2] The positive analysis will examine how directed search in the first stage regulates price dispersion and search effort, and how the equilibrium responds to changes in the meeting efficiency. The normative analysis will investigate when the equilibrium is socially inefficient, what policies can correct the inefficiency, and how these efficient policies should respond to economic conditions.

Section 2 constructs the baseline model of SMS. There is a unit measure of homogeneous visitors, each wanting to consume one indivisible unit of a good. On the other side of the market are homogeneous stayers each wanting to sell a good. Stayers enter the market competitively under a fixed cost of entry and they commit to posted prices to sell the good. In the meeting stage, individuals choose a submarket (p_H, x) to participate, where p_H is the highest price and x the meeting rate for a visitor in the submarket. Individuals take into account the relationship between p_H and x. In each submarket, a meeting function with constant returns to scale determines the measure of meetings. A visitor receives meetings, potentially multiple ones, at the rate x according to the Poisson process. For simplicity, I assume that a stayer receives at most one meeting in the period, and so the meeting process is one-to-many. *Meeting congestion* exists in the sense that the meeting rates on both sides of the market vary with the market tightness. In the second stage, at the end of a period, a visitor chooses which meeting to accept for trade, i.e., to form a match. With homogeneous individuals on each side of the market, the equilibrium has only one active submarket. Prices are continuously distributed in an interval, all of which generate the same expected surplus to a stayer. The meeting rate x is consistent with the two sides' choices of participation in the submarket.

The highest price in a submarket, p_H, plays a unique role. As explained at the end of section 2.1, a stayer would want to increase p_H after entering a submarket. Commitment to p_H is necessary for directing search by providing visitors with the tradeoff between

[2] Some SMS markets, such as the housing market, may have bargaining or bidding in the second stage. I abstract from bargaining or bidding in order to focus on the role of posted prices in directing search.

the meeting probability and the price across submarkets. Other prices do not serve this purpose and do not need commitment. Given p_H, the entire distribution of posted prices arises from rational expectations that a stayer should obtain the same expected profit at all such prices in the submarket. In this sense, the relationship between p_H and x is also sufficient for directing search into submarkets.

The sequentially mixed search equilibrium (SMSE) exists and is unique. An SMSE differs importantly from purely noisy matching, as detailed in section 3. First, the choice of the submarket regulates the price distribution by determining x. Second, a match yields strictly positive surpluses to both sides. In particular, p_H is lower than the monopoly price y that sets a visitor's surplus to zero, in contrast to $p_H = y$ in BJ. Both directed search and meeting congestion are necessary for $p_H < y$. Third, the SMSE is constrained efficient in the baseline model, despite the existence of noisy matching in SMS.

Moreover, the model generates interesting comparative statics: An increase in the meeting efficiency widens price dispersion. This effect arises precisely because visitors respond to the higher meeting efficiency by choosing to enter a submarket that has a higher meeting rate x. The higher x reduces a stayer's matching probability by more at high prices than at low prices. Because all prices must yield the same expected surplus to a stayer, prices fall by less at high levels than at low levels, resulting in wider dispersion. This result helps explain the empirical puzzle that the Internet has not reduced price dispersion, e.g., Baye et al. (2004), Ellison and Ellison (2005).

Section 4 endogenizes visitors' search effort by letting each visitor choose search effort after entering a submarket. In most models, search effort is strategic substitutes among visitors, because one's high search effort increases congestion and reduces the return on search to other visitors. In the current model, search effort can be strategic complements among visitors. The key to this unusual effect is, again, the effect of x on price dispersion. When visitors increase search effort, their meeting rate x increases. This widens price dispersion, which increases the return on search to other visitors. Despite the complementarity in search effort, a unique SMSE exists. The usual congestion from high search effort still exists in the current model to make search effort weak complements among visitors. In the SMSE, a reduction in search cost widens price dispersion, similar to an increase in the meeting efficiency.

I prove that the SMSE with search effort is socially inefficient and prescribe policies for restoring efficiency. Relative to the constrained social optimum, the SMSE has excessive search effort, an inefficiently high meeting rate for a visitor, a deficient meeting rate per search effort for a visitor, and excessive aggregate output. Also, the SMSE has inefficiently wide dispersion in prices, although price levels are lower in the SMSE than in the social optimum. The government can restore social efficiency of the equilibrium by taxing the joint value of a trade and subsidizing stayers' entry cost, both proportionally, and balance the budget by a lump-sum rebate to visitors. Under a mild condition, the corrective policies should lean against the wind, in the sense that the tax on the joint value of trade should increase in an economic boom and decrease in a recession. Note that this policy recommendation has nothing to do with increasing returns to scale in the matching function that Diamond (1982) emphasized. Price dispersion is necessary for the inefficiency here, because the desire to find lower prices is the cause of inefficiently high search effort.

This paper makes several contributions to the literature. First, it constructs a tractable framework for modeling an important phenomenon, SMS. The framework is simple and the mathematics is not hard. All this is deliberate to focus on the novel interaction between the two stages of SMS. The framework can serve as a basis of incorporating other features such as heterogeneity, private information, and many-to-many meetings (see section 5).[3] Second, the paper explains the puzzling effect of the meeting efficiency on price dispersion. Third, the paper shows that search effort can be strategic complements, proves social inefficiency of the SMSE with endogenous search effort, and prescribes corrective policies.

The simplicity of the model might lead one to think that this paper is a routine exercise of grafting directed search onto the BJ model of noisy search. It is not. This should be

[3] Bethune et al. (2018) introduce noisy search into the directed search submarket, but they take the meeting rate as exogenous instead of modeling it as part of the choice of directed search. Kennes et al. (2018) examine three directed search models, one of which (buyer posting) has the flavor of SMS. However, they do not focus on SMS and they abstract from congestion in meetings for visitors (see section 3.2).

obvious from the fact that the BJ model has existed for a long time and yet the literature has not managed to analyze SMS. In fact, grafting directed search onto BJ would not work. In BJ, $p_H = y$, and so all aspects of the price distribution are pinned down by a stayer's indifference among posted prices. There is nothing left for directing search![4] What then makes the modeling of SMS possible in this paper? It is meeting congestion implied by a general meeting function. By generating the tradeoff between the meeting rate and p_H, meeting congestion induces $p_H < y$ and enables p_H to serve as an instrument for directing search. As explained in section 3.2, meeting congestion does not exist in BJ for at least one side of the market and even in auction models cited below that explicitly distinguish meetings from matches. The absence causes $p_H = y$ in those models. Hence, another contribution of this paper is to bring meeting congestion to the forefront, in addition to matching frictions.

The importance of meeting congestions also clarifies the role of the reserve price in competing auctions. Peters and Severinov (1997) construct a model of competing auctions in which sellers use the reserve price to direct search. Albrecht et al. (2012) show that the reserve price does not direct search because it must be equal to a seller's outside option which is exogenous. My analysis demonstrates that if meeting congestions exist on both sides of the market, the reserve price is in the interior between the two sides' outside options and, hence, can serve the role of directing search. See section 3.2 for the exposition.

Some papers have allowed stayers to direct visitors' search with mechanisms instead of prices, e.g., auctions with reserve prices (Julien et al., 2000), potentially vague messages (Menzio, 2007), list prices with bargaining (Stacey, 2015) or with sequential inspection (Lester et al., 2017), and auctions with cheap talk (Kim and Kircher, 2015). SMS differs from auctions in several aspects. First, prices are posted in advance rather than being determined ex post by auctions. Second, visitors are directed into a submarket, not to any particular auction. Third, there is meeting congestion. In the auction models, in contrast, every bidder submits a bid to some auction with probability one.

[4] In addition, grafting directed search onto BJ would face an intractable integer choice because the number of meetings is the primary characteristic there. I model the choice as the continuous variable x, which also enables me to incorporate a general meeting function.

SMS also differs substantially from directed search with multiple applications, e.g., Albrecht et al. (2006) and Galenianos and Kircher (2009). In the latter models, each visitor directs search toward a particular set of stayers and, hence, knows exactly the price or the pricing mechanism that he/she will get if there is a match. In contrast, the prices a visitor encounters in SMS are random draws from the equilibrium distribution. Besides realism, noisy matching in the second stage of SMS affects the equilibrium. In Galenianos and Kircher (2009), the number of equilibrium prices is equal to the number of applications that a worker can send out simultaneously. With SMS, the support of equilibrium prices is a connected interval. Also, the equilibrium in Galenianos and Kircher (2009) is not constrained efficient, but the SMSE is constrained efficient in the baseline model here. Inefficiency of the SMSE arises, instead, when search effort is endogenous.

2. Baseline Model

2.1. The model environment

Consider a one-period economy with a continuum of homogeneous individuals on each side of the market. On one side are homogeneous *stayers* whose measure is elastically determined by competitive entry at the entry cost $k > 0$. The cost of producing a good is normalized to 0. On the other side are *visitors* whose measure is relatively inelastic and, for simplicity, fixed at unity. A visitor wants to consume one unit of an indivisible good and a stayer can produce one unit. Stayers post prices and visitors search. In a trade, the joint value is y, the stayer obtains a surplus p, and the visitor gets $(y - p)$, where p is the price. If an individual fails to trade, the value is 0.[5]

Search is sequentially mixed (SMS). In the first stage, individuals choose which submarket (p_H, x) to enter, expecting the meeting for a visitor, x, to depend on the highest price, p_H. By choosing submarket (p_H, x), stayers commit to posting prices not exceeding

[5] In a directed search environment, Shi and Delacroix (2018) address the question which side should incur the cost to organize trade in order to maximize social welfare in an economy with search frictions. In the current model, the entry cost k is the elastic side's participation cost in the market, while the participation cost of the inelastic side and the cost of creating a trading post are zero. Under these assumptions, the analysis in Shi and Delacroix (2018) implies that social welfare is the same regardless of whether the elastic side or the inelastic side posts prices to organize trade.

p_H but can post any price that honors this commitment. Visitors do not observe the price posted by any specific stayer before meeting the stayer. The second stage is noisy matching in each submarket. Given the measures of stayers and visitors in a submarket, a meeting function, described below, delivers the meeting rate x for a visitor. A visitor receives meetings according to the Poisson distribution and, at the end of the period, chooses which of the received meetings to accept. An accepted meeting is called a *match*, which yields immediate trade at the posted price. For simplicity, I focus on one-to-many meetings by assuming that a stayer receives at most one meeting in the period. See section 5 for a discussion on many-to-many meetings.

In submarket (p_H, x), let N_s be the measure of stayers and N_v the measure of visitors. The meeting function is $M(N_s, N_v)$ $(< N_s)$, which has constant returns to scale. For a visitor, the probability of receiving a number j of meetings in the period is $\frac{x^j e^{-x}}{j!}$, and so the expected number of meetings is x. Since xN_v and $M(N_s, N_v)$ are both equal to the measure of meetings, the equality yields:

$$x = \frac{M(N_s, N_v)}{N_v} = M(\theta, 1), \tag{2.1}$$

where $\theta \equiv \frac{N_s}{N_v}$ is the tightness of the submarket. For any given x, define $\theta(x)$ as the solution for θ to (2.1). This is the tightness needed to deliver the meeting rate x for each visitor in the submarket. A stayer receives at most one meeting in the period and the meeting probability is $\lambda(x) \equiv \frac{M(N_s, N_v)}{N_s} = \frac{x}{\theta(x)}$.

There are two critical features of SMS. First, there are frictions in meetings as well as in matching. Congestion is the common consequence of both frictions. Assumption 1 later will capture meeting congestion by the feature that both meeting rates x and λ vary with θ. Meeting congestion is a realistic feature of many markets. In the labor market, for example, the proper interpretation of a meeting is that an applicant is reviewed or interviewed by a firm. Although a worker can always send an application to a firm, the probability of getting reviewed or interviewed is usually less than one and varies with the labor market tightness. Second, a visitor chooses a trade at the end of the period, rather than whenever a meeting arrives, and so a stayer in a meeting faces potential competitors for the same visitor. As in BJ, noisy, or non-sequential, matching may reflect a fixed cost of making decisions on offers, or simply the inability to decide on an offer on spot. In reality, there may also be cost of waiting for offers. The validity of non-sequential matching is likely to depend on the length of a period. The shorter is a period, the lower is the cost of waiting relative to the fixed cost of making decisions, and the more reasonable is the assumption of non-sequential matching.[6]

In submarket (p_H, x), denote $F(p|x)$ as the cumulative distribution of posted prices, bounded below by p_L and above by p_H. Consider a stayer S in submarket (p_H, x) and let p be the price posted by S. To compute the trading probabilities, I use the following lemma:

Lemma 2.1. *If Z_1 is Poisson with the rate ζ and $Z_2|(Z_1 = j)$ is binomial with (j, q), then Z_2 is Poisson with the rate ζq (see Haight, 1967, p.46). This implies that, conditional on having a meeting, the trading probability for stayer S is $e^{-xF(p|x)}$. For a visitor, the probability of trading at a price no higher than p is $\left[1 - e^{-xF(p|x)}\right]$.*

Let me explain the lemma briefly. Suppose that stayer S met a visitor V. Let j be the number of visitor V's meetings with stayers other than S, and n the number of such meetings with prices no higher than p. Because meetings arrive to V independently from different stayers, j is Poisson distributed with the rate x. In each of those meetings, the price is not higher than p with the probability $F(p|x)$. Conditional on j, n is binomial with $(j, F(p|x))$, and so $\Pr(n = 0|j) = [1 - F(p|x)]^j$. Unconditional on j, stayer S expects visitor V to accept the match with the following probability:[7]

$$\Pr(n = 0) = \sum\nolimits_{j=0}^{\infty} \frac{x^j e^{-x}}{j!} [1 - F(p|x)]^j = e^{-xF(p|x)}.$$

This verifies the result for stayer S in Lemma 2.1. Similarly, for a visitor, conditional on the number of meetings, j, the number n of meetings with prices no higher than p is

[6]Menzio and Trachter (2015) construct a model to generate equilibrium price dispersion with sequential matching instead of noisy matching. The important ingredient of their model is the existence of a large player who has positive mass and receives meetings at a different rate than all other stayers who have zero mass and lie in a continuum.

[7]If a visitor receives two or more identical prices that are the lowest among the received offers, the visitor chooses one of them with equal probability. It will become clear later that the offer distribution contains no mass point. Thus, the probability that stayer S faces a competitor who offers the same price and has met the same visitor is zero.

binomial with $(j, F(p|x))$. Unconditional on j, n is Poisson with the rate $xF(p|x)$. Hence, the visitor trades at a price no higher than p with the probability $\Pr(n \geq 1) = 1 - e^{-xF(p|x)}$.

Conditional on having a meeting, a stayer's expected surplus is

$$\Pi(p, x, F) \equiv pe^{-xF(p|x)}. \tag{2.2}$$

Denote the maximum of a stayer's expected surplus in a meeting as

$$\pi(x) = \max_{p \leq p_H} \Pi(p, x, F). \tag{2.3}$$

A stayer's expected profit of entering submarket (p_H, x) is $\lambda(x)\pi(x) - k$, where λ is the stayer's meeting probability in the period and k the entry cost. For a visitor, the expected surplus in the submarket is:

$$D(x) \equiv \int_{p_L}^{p_H} (y - p) \, \mathrm{d}\left[1 - e^{-xF(p|x)}\right], \tag{2.4}$$

where d is with respect to p.

The trading process endogenously generates an aggregate matching function and a distribution of transaction prices. A visitor trades with the probability $1 - e^{-x}$. The measure of matches in the period is $N_v(1 - e^{-x})$. This *matching function* has constant returns to scale. Conditional on having at least one meeting, a visitor succeeds in trading at a price no higher than p with the probability:

$$G(p|x) \equiv \frac{1 - e^{-xF(p|x)}}{1 - e^{-x}}. \tag{2.5}$$

This is the cumulative distribution of transaction prices. It is clear that $G(p|x) \geq F(p|x)$ for all p, with strict inequality for all $p \in (p_L, p_H)$. Search enables visitors to obtain prices lower than posted prices in the first-order stochastic dominance.

A *sequentially mixed search equilibrium (SMSE)* consists of a tightness function, $\theta(x)$, stayers' choice of the submarket to enter and the price to post, visitors' choice of the submarket to enter, a set I of active submarkets, $(p_{Hi}, x_i)_{i \in I}$, and the implied distribution of posted prices, $(F_i)_{i \in I}$, that satisfy (i)-(iii) below:

(i) Optimality of pricing decisions: For each $i \in I$, a stayer in submarket i optimally chooses the price p to post to maximize $\Pi(p, x_i, F_i)$ under the constraint $p \leq p_{Hi}$. All posted prices generate $\pi(x_i)$ as the maximized Π and they constitute the distribution F_i.
(ii) Optimality of search: Given $\theta(.)$, a visitor chooses the submarket to enter to maximize D subject to the dependence of F on x and, especially, the dependence of p_H on x. The set of submarkets that receive positive measures of visitors is I.
(iii) The tightness function: $\theta(x)$ induces $\lambda(x)$ that satisfies a complementary slackness condition on competitive entry of stayers. That is, $\lambda(x)\pi(x) = k$ for all submarkets with $\pi(x) \geq k$ and $\theta(x) = 0$ for all submarkets with $\pi(x) < k$.

Requirements (i) and (ii) are for submarkets active in the SMSE. Requirement (iii) is for all submarkets, which restricts the beliefs on the function $\theta(x)$ for both inactive and active submarkets (see Menzio and Shi, 2010, and Guerrieri, et al., 2010). For every active submarket, the expected profit of a stayer should be zero. For every inactive submarket, a stayer should be unwilling to enter the submarket even if the stayer would meet a visitor with probability one. This requirement prevents a submarket from being inactive just because individuals on each side expect no one on the other side will enter the submarket.

Why is the highest price p_H special in the model? Can the lowest price serve equally well for directing search? To answer these questions, it is important to note that the entire distribution of posted prices depends on x. However, given p_H, the distribution of posted prices arises from rational expectations that all prices posted in a submarket should generate the same expected profit for a stayer (see (i) in the equilibrium definition). Thus, under rational expectations, p_H is sufficient for directing search. The commitment to p_H is also necessary for directing search whenever $p_H < y$. If there were no commitment to p_H, a stayer after entering a submarket would profit by deviating to p_H^+ slightly higher than p_H. Once in the submarket, a stayer's meeting probability is independent of the posted price. Moreover, at p_H^+ and p_H, a stayer has the same probability to trade, which occurs when the stayer meets a buyer who has no other meeting. However, a trade at p_H^+ yields higher profit to a stayer than at p_H, which implies that expected profit is higher at p_H^+. Precisely, $\Pi_1\left(p_H^+, x, F\right) > 0$ if $p_H < y$. In contrast, reducing the price from the lowest price

p_L reduces a stayer's expected profit, and increasing the price from p_L does not change a stayer's expected profit. Thus, the lowest price, or any price strictly lower than p_H, serves no role in directing search beyond rational expectations on the price distribution.

2.2. Optimal choices and the SMSE

To characterize optimal decisions, define $\bar{x} \leq \infty$ by $\theta(\bar{x}) = \infty$ as the natural upper bound on x, and define the elasticity of θ as $\varepsilon(x) \equiv \frac{x\theta'(x)}{\theta(x)}$.

Assumption 1. *The meeting function is such that* $\lambda(x) \in (0,1)$, $\theta'(x) > 0$ *and* $\theta''(x) > 0$ *for all* $x \in (0, \bar{x})$, *and that* $\theta(0) = 0$, $\theta'(0) > 0$, *and* $\theta'(\bar{x}) = \infty$. *Also,* $\varepsilon(x) > 1$ *for all* $x > 0$, *and* $\theta'(0)k < y$.

The assumption $\lambda(x) < 1$ says that a stayer receives at most one meeting. The assumptions on $\theta(x)$ are equivalent to the standard properties: M is strictly increasing and concave in each of the two arguments, $M(0,1) = 0$, $M_1(0,1) < \infty$ and $M_1(\infty,1) = 0$.[8] The assumptions on $\theta(x)$ imply $\lambda'(x) < 0$ and $\lambda(\bar{x}) = 0$. The feature $\varepsilon(x) > 1$ is listed for a reference. It is not an additional feature but rather an implication of $\lambda'(x) < 0$, because $\varepsilon = 1 - \theta\lambda'$. The assumption $\theta'(0)k < y$ requires the joint surplus of a trade to exceed the cost of stayers' entry when the measure of stayers in the market is arbitrarily small. If this assumption is violated, it is socially efficient to shut down the market. The following well-known meeting functions satisfy Assumption 1 with $k < y$:

Example 2.2. *The Dagum (1975) function is* $M(\theta,1) = (\theta^{-\rho} + 1)^{-1/\rho}$ *with* $\rho \in (0,\infty)$. *This function yields* $\theta(x) = [x^{-\rho} - 1]^{-1/\rho}$ *and* $\varepsilon(x) = [1 - x^{\rho}]^{-1}$. *Another function is the urn-ball function* $M(\theta,1) = \theta\left(1 - e^{-1/\theta}\right)$. *This function yields* $\varepsilon(x) = \frac{e^{1/\theta(x)} - 1}{e^{1/\theta(x)} - 1 - \frac{1}{\theta(x)}}$. *With both functions,* $\lambda(0) = \bar{x} = 1$, $\theta'(0) = 1$, *and* $\varepsilon'(x) > 0$.

The assumptions on $\theta(x)$ capture meeting congestion on the two sides of the market. That is, x increases, and λ decreases, in the market tightness θ. An increase in the

[8] Because $M(N_s, N_v) = N_v M(\theta,1)$, then $M_2(N_s, N_v) = x - \theta M_1(\theta,1)$. The property $M_2(N_s, N_v) > 0$ and the fact $M_1(\theta,1) = 1/\theta'(x)$ imply $\varepsilon(x) > 1$.

market tightness alleviates congestion of visitors but exacerbates congestion of stayers in the meeting stage. It is useful to mention that meeting congestion is absent on one side of the market in the urn-ball process, e.g., Burdett et al. (2001) and Julien et al. (2000). In this process, a "ball" (visitor) is thrown at an "urn" (stayer) in the meeting stage and then each urn selects one received ball. A visitor's meeting probability (one) is independent of the market tightness.[9]

To analyze optimal choices, consider a stayer first. The individual makes two decisions sequentially: which submarket to enter and what price to post in the submarket. Suppose that the stayer has already chosen to enter submarket (p_H, x). The stayer chooses the posted price $p \leq p_H$ to maximize the expected surplus in a meeting, $\Pi(p, x, F)$, as in (2.2). The maximized expected surplus is $\pi(x)$. If a price $p \leq p_H$ is not in the support of F, then it must be the case that $\Pi(p, x, F) < \pi(x)$. Conversely, all prices in the support of F must yield the same expected payoff to a stayer in a meeting as $\pi(x)$. That is,

$$\Pi(p, x, F) = \pi(x) \text{ for all } p \in \text{ supp}(F).$$

This equal-surplus condition determines the distribution of offers in the submarket:

$$F(p|x) = \frac{1}{x} \ln\left(\frac{p}{\pi(x)}\right) \text{ for all } p \in \text{ supp}(F). \tag{2.6}$$

It is convenient to invert F as

$$p = \pi(x) e^{xF} \text{ for all } p \in \text{ supp}(F). \tag{2.7}$$

Because the bounds on the support must satisfy $F(p_L) = 0$ and $F(p_H) = 1$, then[10]

$$p_L = \pi, \quad p_H = \pi(x) e^{x}. \tag{2.8}$$

The formula for p_H in (2.8) is the relationship between x and p_H needed for directing the choice of the submarket. A stayer's expected profit of entering submarket (p_H, x) is

[9]This absence of meeting congestion generalizes to the case where a visitor reaches some stayer with a constant probability $a \leq 1$.

[10]The price p_H should satisfy a visitor's individual rationality: $p_H \leq y$. I will show later in Corollary 3.2 that this inequality holds strictly in the equilibrium.

$\lambda(x)\pi(x) - k$. In every submarket with $\pi(x) < k$, the tightness is $\theta(x) = 0$. In every submarket with $\pi(x) \geq k$, a stayer's expected profit of entry is zero. This yields:

$$\pi(x) = \frac{k}{\lambda(x)}. \tag{2.9}$$

Now consider a visitor. If the visitor participates in submarket (p_H, x), the expected surplus is $D(x)$, as given by (2.4). The visitor chooses x to maximize $D(x)$, taking the relationships (2.6) and (2.9) into account. Substituting p from (2.7) into (2.4), changing the integration variable from p to F and then substituting π from (2.9), I have:

$$D(x) = y\left(1 - e^{-x}\right) - k\theta(x). \tag{2.10}$$

For future references, it is useful to interpret the term $k\theta(x)$. This term is equal to $\pi(x)x$, after using (2.9) and $\lambda(x) = x/\theta(x)$. The term is the expected surplus that a visitor concedes to a stayer in the submarket. The appearance of the product between π and x should not be construed as suggesting that a visitor can have more than one trade. Rather, a visitor has at most one trade in the period. However, a stayer "prices in" the risk of losing a visitor to a competitor. The potential failure to trade with a visitor in a meeting raises the price that the stayer charges in order to break even for entry, resulting in the expected amount $\pi(x)x$ being taken away from a visitor.

Because $\theta(x)$ is a strictly convex function, $D(x)$ is strictly concave and, hence, is maximized by a unique x. Denote the unique maximizer as x^*. If x^* is interior, then it satisfies the first-order condition:

$$\theta'(x^*)e^{x^*} = \frac{y}{k}. \tag{2.11}$$

It is easy to verify $x^* < \bar{x}$, where $\theta(\bar{x}) = \infty$. Also, $x^* > 0$ under the maintained assumption $y > \theta'(0)k$. The above analysis has established the following proposition:

Proposition 2.3. *A unique SMSE exists, with only submarket (p_H, x^*) being active, where $x^* \in (0, \bar{x})$ solves (2.11). Posted prices are continuously distributed according to F^* over $[p_L, p_H]$, where p_L and p_H are given by (2.8) and F^* by (2.6) with $x = x^*$. Transaction prices are distributed according to G in (2.5).*

Homogeneity among visitors is clearly responsible for the SMSE to have only one active submarket. If there are n types of visitors who differ in the valuation, then the SMSE will have n active submarkets (see section 5).

3. Equilibrium Properties, Efficiency and Interpretations

3.1. Directed search regulates the price distribution

The current model encompasses two classes of search models as special cases. Purely directed search is the special case where each visitor is restricted to having at most one meeting, e.g., Peters (1991), Burdett et al. (2001) and Julien et al. (2000). Purely noisy search is the special case where x is exogenous and there is no commitment to p_H, e.g., the BJ model. Because noisy matching is the common cause of price dispersion in the current model and in BJ, I focus on the link and the contrast between the two.

Noisy matching with multiple offers induces a continuous distribution of prices in an interval.[11] A stayer may lose the visitor to a competitor. To increase the trading probability, a stayer would want to post a price slightly below the competitor's if he/she knew what price a competitor posts. This implies that the price distribution cannot have a mass point anywhere above the lowest price p_L. If there were, posting a slightly lower price would increase a visitor's acceptance probability by a discrete amount and, hence, would increase the stayer's expected surplus. There cannot be a mass point at the lowest price p_L, either. If there were, then $p_L = 0$, because any $p_L > 0$ would make it profitable for a stayer to undercut the price. But if $p_L = 0$, then posting a price $p > p_L$ would yield a higher (positive) expected profit than posting p_L. In the equilibrium, prices are continuously distributed. In addition, the support of the price distribution must be a connected interval. If there were a "hole" in the support, a stayer posting a price inside the hole could increase the ex post gain without reducing the acceptance probability, relative to posting the price at the lower boundary of the hole.

The current model differs from BJ primarily in the presence of directed search in SMS.

[11] BJ model noisy matching slightly differently and do not impose the unit upper bound on a stayer's capacity. These additional differences are not essential for the price distribution to be non-degenerate.

The BJ model, phrased in the notation of the current model, fixes x and considers a market with a fixed measure of individuals on each side of the market. In an extension, BJ add a cost of x to endogenize visitors' choice of x, but keep the assumption that search is not directed by any relationship between x and p_H. As explained in the Introduction and further below, $p_H = y$ in BJ, and so one cannot extend BJ to incorporate directed search.

Directed search in the first stage of SMS affects the equilibrium substantially by determining the optimal choice of x. The equilibrium has $x^* \in (0, \infty)$ (see Proposition 2.3), and so the probability that a visitor receives two or more meetings, $1 - (1 + x^*) e^{-x^*}$, is strictly bounded in $(0, 1)$. This interior probability is critical for the equilibrium price distribution to be non-degenerate. Moreover, through x^*, directed search affects the entire distribution of prices in the equilibrium. To state these effects, denote the *price spread* as $\Delta p \equiv p_H - p_L$ and the coefficient of variation in posted prices as $cv_F \equiv \frac{[var_F]^{1/2}}{\mathbb{E}_F}$, where $\mathbb{E}_F$ is the mean and var_F the variance of posted prices. The coefficient of variation in transaction prices is cv_G, where the distribution G in (2.5) is used instead of F. Computation yields:

$$cv_F(x) = \left[\frac{x(e^x + 1)}{2(e^x - 1)} - 1\right]^{\frac{1}{2}}, \quad cv_G(x) = \left[e^x \left(\frac{1 - e^{-x}}{x}\right)^2 - 1\right]^{1/2}. \tag{3.1}$$

These coefficients of variation are functions of only x, with no parameters. Any change in a parameter of the model can change cv_F and cv_G only if it affects x. For this reason, it is useful to examine this effect of x on price dispersion, even though x is endogenous. The proof of the following corollary is straightforward and omitted:

Corollary 3.1. *An increase in x increases posted and transaction prices in the first-order stochastic dominance in F and G. Also, an increase in x increases the bounds (p_L, p_H), the price spread Δp, and the coefficients of variation (cv_F, cv_G).*

For a stayer, an increase in x reduces both the meeting probability and the matching probability conditional on having a meeting. Both reduce a stayer's expected profit. To compensate for a stayer's entry cost, posted and transaction prices must be higher. The positive relationship between posted prices and x is important for directed search. A submarket with a higher meeting rate for a visitor necessarily has higher prices. The optimal tradeoff between the meeting rate and prices yields the equilibrium x.

For an increase in x to affect price dispersion, it must affect stayers differently depending on their posted prices. This non-uniform effect arises from the effect of x on a stayer's matching instead of meeting probability. Conditional on having a meeting, a stayer is less likely to succeed in a trade if the posted price is higher. As a result, an increase in x reduces a stayer's trading probability by more if the stayer posts a higher price. To yield the same expected surplus conditional on having a meeting, π, the price must increase by more if the price is initially higher. This is why an increase in x increases the bounds on prices, the price spread and the coefficient of variation in prices.

3.2. Every trade generates strictly positive surpluses for both sides

If $p_L > 0$, a stayer obtains a strictly positive surplus in a trade. If $p_H < y$, a visitor obtains a strictly positive surplus in a trade. The proof of the following corollary can be constructed from the ensuing explanation:

Corollary 3.2. *$p_H < y$ and $p_L > 0$, and so every trade generates strictly positive surpluses for both sides of the market.*

The result $p_L > 0$ is evident, because posting price 0 implies zero expected profit. The result $p_H < y$ illustrates the importance of directed search in SMS. When search is purely noisy, as in BJ, stayers do not commit to any feature of the price distribution. In this case, for any $p_H < y$, it is profitable for a stayer to increase p_H, as explained at the end of section 2.1. This is why the equilibrium in BJ has $p_H = y$. With directed search, in contrast, visitors choose the submarket to participate. In the submarket with $p_H = y$, prices may be too high to be optimal for the tradeoff between the trading probability and the trade surplus. Catering to visitors' tradeoff, stayers may optimally set $p_H < y$.

Although directed search is necessary for $p_H < y$, it is *not sufficient*. To find other necessary ingredients for $p_H < y$, let me explain how the result arises in this model by considering a marginal increase in p_H. The higher price induces more stayers to enter the submarket, which generates an increase in a visitor's meeting rate x according to $p_H = \pi(x) e^x$ (see (2.8)). A higher x increases a visitor's trading probability by e^{-x} and, hence, increases the expected surplus of a meeting by ye^{-x}. To compensate stayers for entering the submarket to support the higher x, a visitor must increase the expected surplus conceded to a stayer by $k\theta'(x)$, as explained for (2.10). If p_H is optimal, then the net benefit of the higher x to a visitor must be equal to zero. That is, $ye^{-x} = k\theta'(x)$ (see (2.11)). This is consistent with the relationship $p_H = \pi(x) e^x$ if and only if

$$\frac{p_H}{y} = \frac{\pi(x)}{k\theta'(x)} = \frac{1}{\varepsilon(x)},$$

where the second equality follows from substituting $\pi(x) = k/\lambda(x)$ and $\lambda(x) = x/\theta(x)$. Because $\varepsilon(x) > 1$ by Assumption 1, then $p_H < y$.

The above explanation shows that, in addition to directed search, another necessary condition for $p_H < y$ is $\varepsilon(x) > 1$. This condition captures meeting congestion on the two sides of the market, as explained for Assumption 1. There are well-known examples where at least one side of the market does not face meeting congestion. One example is an economy where the meeting probability is one for the short side of the market; i.e., $x = \min\{\theta, 1\}$ and $\lambda = \min\{1, \frac{1}{\theta}\}$. If $\theta < 1$, then $\lambda = 1$. If $\theta \geq 1$, then $x = 1$. In both cases, $\varepsilon(x) = 1$ and $p_H = y$. Another example is the urn-ball process discussed after Assumption 1. In this process, x is a constant (one), which yields $\varepsilon(x) = 1$ and $p_H = y$. In these examples, either a visitor's meeting rate x or a stayer's meeting rate λ is independent of the market tightness, which causes $p_H = y$ even if search is directed.

The general interpretation of the outcome $p_H = y$ is that, if one side of the market faces meeting congestions but the other side does not, the side that faces meeting congestions should have zero surplus at the lowest equilibrium price.. Under this general interpretation, the result $p_H = y$ holds in some auction models with directed search. For example, in Julien et al. (2000), workers post auctions to direct firms' search. In one of the models in Kennes et al. (2018), firms post auctions to direct workers' search. In both models, the search process is urn-ball and so a bidder meets one of the auctions with certainty. Facing no congestion in the meeting stage, a bidder faces no tradeoff between the meeting probability and the price. It is then optimal for an auctioneer to attract bidders to participate in his/her auction by setting his/her own surplus to zero at the reserve price.[12] In this case, since the reserve price is equal to an exogenous number, it is unable to direct search. This is the correction that Albrecht et al. (2012) have made to the result in Peters and Severinov (1997). Corollary 3.2 indicates that if both sides of the market face meeting congestions, both sides have a strictly positive surplus at the reserve price, and so the reserve price resumes the role of directing search in competing auctions.

3.3. Social efficiency of the SMSE

Another main difference of SMSE from an equilibrium with purely noisy search is constrained social efficiency. To analyze efficiency, consider a social planner who maximizes social welfare measured by the sum of expected surpluses in the economy. The planner chooses the measure of stayers to enter the market, N_s, but is constrained by the same frictions as in the market. Specifically, the planner takes as given the meeting function $M(N_s, N_v)$. If a visitor receives multiple meetings, the planner randomly chooses one of them for the visitor to trade. Total cost of stayers' entry is kN_s, and the measure of trades is $N_v(1-e^{-x})$. Since a trade generates the joint surplus y, social welfare is:

$$yN_v\left(1-e^{-x}\right)-kN_s.$$

To maximize social welfare, the planner chooses N_s subject to $x=\frac{1}{N_v}M(N_s,N_v)$. Because N_v is fixed, the planner's choice of N_s is equivalent to the choice of x. Dividing social welfare by N_v, normalized welfare (per visitor) is:

$$y\left(1-e^{-x}\right)-k\theta(x),$$

where $\theta=\frac{N_s}{N_v}$ is used. This welfare measure is the same as $D(x)$ in (2.10) – a visitor's expected surplus of entering a submarket (p_H, x). Thus, the planner's choice of x is identical to a visitor's choice in the SMSE. This proves the following proposition:

[12]This is true when the market is large. If the market is finite, the matching function has decreasing returns to scale, and an auctioneer's surplus is positive at the reserve price. See Julien et al. (2000).

Proposition 3.3. *The SMSE is constrained efficient.*

Directed search in SMS and competitive entry of stayers are important for the equilibrium to be socially efficient, as emphasized in the directed search literature (e.g., Moen, 1997, Shi, 2001). The above proposition adds to the literature by showing that the equilibrium is efficient even if the search process has noisy matching in the second stage. In contrast, if search is purely noisy, then the equilibrium is socially inefficient. In this case, $p_H = y$, which is inefficiently higher and induces excessive entry of stayers.

There is a tight connection between a visitor's optimality condition, (2.11), and the social planner's choice. When choosing the meeting rate for a visitor, the planner equates the social marginal benefit of a higher x to the social marginal cost. The social marginal benefit of a higher x is that it reduces the visitor's probability of failing to trade. Since the failure probability is e^{-x}, a marginally higher x reduces this probability by e^{-x}. The expected social gain is $e^{-x} y N_v$. The marginal cost of increasing x is that more stayers need to enter the market to deliver the higher x. To increase x marginally, θ must increase by $\theta'(x)$, and so the measure of stayers must increase by $\theta'(x) N_v$. Thus, the entry cost increases in an amount $k\theta'(x) N_v$. The equilibrium condition (2.11) equates this social marginal cost to the social marginal benefit of a higher x.

3.4. Effects of the meeting efficiency

The Internet has increased the convenience of trading, but it has not reduced price dispersion significantly. Baye et al. (2004) compared the online market with the offline market for consumer electronics and found similar price dispersion in the two markets. Ellison and Ellison (2005) found a similar puzzle in other online markets. To resolve this puzzle, I model the increasing use of the Internet as an increase in the meeting efficiency. Section 4.1 will re-examine the issue as a reduction in search cost.

Let the meeting function be $AM(N_s, N_v)$, where A is the meeting efficiency. Denote $\nu = \frac{x}{A}$. The definition of x implies $\nu = M(\theta, 1)$, which solves θ as $\theta(\nu)$. The elasticity of this tightness function is $\varepsilon(\nu) = \frac{\nu\theta'(\nu)}{\theta(\nu)}$. The meeting probability for a stayer is $\lambda = \frac{x}{\theta(\nu)}$.

Except the change in the argument of θ and ε from x to ν, the SMSE is characterized as before. Modifying (2.11), the first-order condition for the optimal x^* is:

$$\frac{1}{A}\theta'(\frac{x^*}{A})e^{x^*} = \frac{y}{k}. \tag{3.2}$$

Define $A_0 \in (0, \infty)$ as the unique solution to:

$$\frac{e}{A_0}\theta'(\frac{1}{A_0}) = \frac{y}{k}. \tag{3.3}$$

Then, $x^* < 1$ if and only if $A < A_0$. The following proposition is proven in the Supplementary Appendix C:

Proposition 3.4. *(i)* $\frac{dx^*}{dA} > 0$ *and* $\frac{d}{dA}cv_i(x) > 0$ *for* $i = F, G$. *(ii)* $\frac{d\theta}{dA} > 0$ *iff* $A < A_0$. *(iii)* $\frac{dp_H}{dA} \leq 0$ *iff* $(A_0 - A)\varepsilon' > 0$*; If* $\varepsilon' \geq 0$ *then* $\frac{dp_L}{dA} < 0$ *and* $\frac{d\Delta p}{dA} > 0$. *(iv) If* $\varepsilon' \geq 0$ *and* $A < A_0$*, an increase in* A *reduces posted and transaction prices in the first-order stochastic dominance.*

An increase in the meeting efficiency unambiguously widens price dispersion measured by the coefficient of variation, although price levels may change ambiguously. By increasing stayers' meeting rate, an increase in A has direct effects of increasing entry of stayers and depressing price levels. However, an increase in A also increases a visitor's meeting rate x, which increases prices and widens price dispersion (see Corollary 3.1). Specifically, an increase in x reduces a stayer's matching probability by more at high prices than at low prices. To keep a stayer indifferent between posting different prices, a stayer's match surplus conditional on having a trade must fall by less at high prices than at low prices. If prices fall after the increase in A, they must fall by less at high levels than at low levels. If prices rise after the increase in A, they must rise by more at high levels than at low levels. In both cases, price dispersion widens.

In the empirical evidence cited above, the increased use of the Internet did not widen price dispersion significantly, although it did not compress price dispersion either. To reconcile with this evidence, one may recognize that the Internet can reduce a stayer's entry k, in addition to increasing A. It can be verified that a reduction in k increases the measure of stayers, reduces x and compresses price dispersion. Thus, the combined effect of an increase in A and a reduction in k may leave price dispersion unchanged.

Proposition 3.4 contains additional information about the effects of an increase in A. First, an increase in A may increase or decrease the market tightness. If the submarket does not change, an increase in A increases the meeting rate for stayers, which induces more stayers to enter the market to increase the market tightness. However, an increases in A also induces visitors to choose a submarket with a higher x, which reduces the matching rate for a stayer and, hence, reduces stayers' entry. The direct effect of A on a stayer's matching rate dominates if $x < 1$, which is equivalent to $A < A_0$. In this case, an increase in A increases stayers' entry to increase the market tightness.

Second, an increase in A reduces the lowest price and widens the price spread if and only if $\varepsilon' \geq 0$. Because $\varepsilon(\nu) = \frac{\nu\theta'(\nu)}{\theta(\nu)}$ is the elasticity of θ to x, the derivative ε' measures the sensitivity of the market tightness to x. The case $\varepsilon' \geq 0$ is the normal case, as shown later with examples. If $\varepsilon' \geq 0$, then an increase in x causes the market tightness to increase sharply by increasing stayers' entry. Such a large entry of stayers depresses prices. This negative effect on prices is stronger at low prices than at high prices, as explained for Corollary 3.1 and repeated above. Thus, if $\varepsilon' \geq 0$, the lowest price falls and the price spread widens. The highest price also falls if the market tightness indeed increases, i.e., if $A < A_0$. Conversely, if ε' is sufficiently negative, the market tightness is insensitive to changes in x. In this case, prices must rise in order to induce more stayers to enter the market to deliver the increase in x, and the price spread narrows. Note that $\varepsilon' \geq 0$ is satisfied by the well-known meeting functions in Example 2.2.

Third, when $A < A_0$ and $\varepsilon' \geq 0$, an increase in the meeting efficiency reduces all prices according to the distribution. This is the case where the market tightness increases in A and is sensitive to x. In this case, an increase in A induces a large increase in stayers' entry into the market and, hence, reduces prices. Despite lower prices, the increase in A compensates stayers' entry into the market by increasing a stayer's meeting probability.

4. Endogenous Search Effort and Efficient Policies

This section endogenizes visitors' search effort. I show that price dispersion can induce search effort to be strategic complements among visitors and the equilibrium to be socially inefficient. I analyze the effects of a reduction in search cost and examine policies for efficiently managing aggregate activities.

4.1. Equilibrium with endogenous search effort

A visitor chooses search effort, s, after entering a submarket. Let $\psi(s)$ be the cost of search effort, with $\psi' > 0$ and $\psi'' > 0$ for all $s > 0$, and $\psi'(0) = 0$. Let the meeting rate for a visitor be $x = (s + s_0) z$, where z denotes the meeting rate per search effort. The constant $s_0 > 0$ rules out the uninteresting case where the equilibrium can be stuck at $z = s = 0$.[13] The total measure of visitors' search effort in a submarket is $(s + s_0) N_v$, and the measure of meetings in the submarket is $M(N_s, (s + s_0) N_v)$. Since the total measure of meetings in the period is also equal to xN_v, then

$$z = \frac{M(N_s, (s + s_0) N_v)}{(s + s_0) N_v} = M(\theta, 1),$$

where $\theta = \frac{N_s}{(s+s_0)N_v}$ is the effective tightness of the submarket. The above equation solves $\theta = \theta(z)$, which is a function of z instead of x. The meeting probability for a stayer in a period is $\lambda(z) = \frac{z}{\theta(z)}$. Define $\bar{z} \leq \infty$ by $\theta(\bar{z}) = \infty$. Submarkets are now described by (p_H, z), instead of (p_H, x). As a result, competitive entry of stayers into the market yields a stayer's expected surplus conditional on having a meeting as $\pi(z) = \frac{k}{\lambda(z)}$, which is a function of z instead of x.

For stayers, the formulas in the baseline model remain valid after replacing $\pi(x)$ by $\pi(z)$ and $F(p|x)$ by $F(p|z)$. In a meeting, a stayer posting p succeeds in trade with the probability $e^{-xF(p|z)}$. The stayer's expected surplus in the meeting is $pe^{-xF(p|z)}$. Equating

[13] If $s_0 = 0$, a visitor's meeting rate is sz, in which case the marginal gain to increasing s is zero if $z = 0$ and the marginal gain to increasing z is also zero if $s = 0$. Even a small $s_0 > 0$ can prevent this uninteresting case from happening in the equilibrium.

this expected surplus to $\pi(z)$ for all prices on the support of F yields:

$$F(p|z) = \frac{1}{x} \ln \frac{p}{\pi(z)}. \tag{4.1}$$

The inverse of this distribution function is:

$$p = \pi(z) e^{xF}. \tag{4.2}$$

The bounds on the support of F are:

$$p_L = \pi(z), \quad p_H = \pi(z) e^{x}. \tag{4.3}$$

To analyze a visitor's optimal choice, suppose that one visitor searches with effort $\tilde{s}$ while all other visitors search with effort s. For the deviating visitor, the meeting rate is $\tilde{x} \equiv (\tilde{s} + s_0) z$, and the probability of trading at a price no higher than p is $1 - e^{-\tilde{x}F(p|z)}$. The deviating visitor's expected surplus in the submarket is:

$$D(z, \tilde{s}) \equiv -\psi(\tilde{s}) + \int_{p_L}^{p_H} (y - p) \, \mathrm{d}\left[1 - e^{-\tilde{x}F(p|z)}\right]. \tag{4.4}$$

This modifies (2.4) by subtracting search cost and changing the meeting rate to $\tilde{x}$. Note that stayers make their decisions based on the expectation that almost all visitors in the submarket will search with effort s. That is, p is given by (4.2) where search effort is s instead of $\tilde{s}$. Substituting such p, integrating and substituting π from (2.9), I have:

$$D(z, \tilde{s}) = -\psi(\tilde{s}) + y\left[1 - e^{-\tilde{x}}\right] - k\theta(z)(\tilde{s} + s_0) L((\tilde{s} - s) z), \tag{4.5}$$

where $\tilde{x} = (\tilde{s} + s_0) z$ and

$$L(t) \equiv \frac{1 - e^{-t}}{t} \text{ for all } |t| < \infty. \tag{4.6}$$

The visitor chooses $(z, \tilde{s})$ to maximize $D(z, \tilde{s})$.

The term $L((\tilde{s} - s) z)$ captures a visitor's incentive to search more intensively than others to find a lower price. To see this, note that the last term in (4.5) is equal to $\tilde{x} L \pi$. As in the explanation for (2.10), this term is the expected surplus that the deviating visitor concedes to a stayer in the submarket. Thus, $L\pi$ is the concession normalized by the expected number of meetings for the visitor. If the visitor chooses the same search effort as other visitors do, this normalized concession is $L(0)\pi = \pi$. However, by choosing $\tilde{s} \neq s$ off the equilibrium path, the visitor can affect the concession. This externality critically depends on the properties of the function L, which are listed in the following lemma (see the Supplementary Appendix C for a proof):

Lemma 4.1. *The function $L(t)$ defined by (4.6) has the following properties: (i) $L(0) = 1$, $L'(0) = -\frac{1}{2}$, and $L''(0) = \frac{1}{3}$; (ii) $L(t) > 0$, $L'(t) < 0$ and $L''(t) > 0$ for all $|t| < \infty$.*

The property $L'(t) < 0$ is the most important one. If the deviating visitor searches more intensively than other visitors do, the visitor expects to be more likely to encounter a meeting with a lower price and, hence, to concede less than π to a stayer in a meeting. Conversely, if the visitor searches less intensively than other visitors, the visitor expects to concede more than π to a stayer in a meeting. Notice that this benefit from higher search effort exists even when $\tilde{s}$ is arbitrarily close to s, because $L'(0) < 0$. The benefit diminishes as the visitor keeps increasing search effort, because $-L'' < 0$.

An SMSE can be defined similarly to that in section 2.1 by adding search effort $\tilde{s}$ to a visitor's choice. I focus on symmetric equilibria, where $\tilde{s} = s$. As shown in the proof of Proposition 4.2 below, the optimal $(z, \tilde{s})$ are interior and satisfy the first-order conditions:

$$0 = ye^{-\tilde{x}} - k\left[\theta' L + (\tilde{s} - s)\,\theta L'\right] \tag{4.7}$$

$$0 = -\psi' + yze^{-\tilde{x}} - k\theta(z)\left[L + \tilde{x}L'\right]. \tag{4.8}$$

Denote the optimal choice of z as z^*. In any symmetric SMSE, these conditions become:

$$\begin{aligned} &\theta'(z) = \tfrac{y}{k}e^{-x}, \\ &\psi'(s) = k\theta(z)\left[\varepsilon(z) - 1 + \tfrac{x}{2}\right]. \end{aligned} \tag{4.9}$$

Recall that $x = (s + s_0)\,z$. Given any s, denote $z_a(s)$ as the solution for z to the first equation in (4.9) and $z_b(s)$ as the solution for z to the second equation. Figure 1 depicts $z_a(s)$ and two possibilities of $z_b(s)$. The SMSE is depicted by points E1 and E2, respectively, for the two possibilities of $z_b(s)$. The SMSE is unique. Formally, for any given (s, z), denote a visitor's the optimal choice of $\tilde{s}$ as $S(s, z)$. The following proposition holds (see Appendix A for a proof):

Proposition 4.2. *A unique symmetric SMSE exists. In the SMSE, $S_1(s, z^*) < 1$. Moreover, $S_1(s, z^*) > 0$ if and only if $x^* \equiv (s + s_0) z^* > \frac{3}{2}$. An exogenous reduction in ψ' reduces z, increases (s, x), and increases (cv_F, cv_G). If $\varepsilon' \geq 0$, this reduction in search cost reduces posted and transaction prices in the first-order stochastic dominance.*

The feature $S_1(s, z^*) > 0$ means that search effort is "strategic complements" among visitors. Higher search effort by some visitors widens price dispersion and, thereby, increases the return on search to other visitors. For any given z, higher search effort by some visitors increases their meeting rates, x, and pushes prices down. This negative effect on prices is stronger at high prices than at low prices, as explained for Corollary 3.1. To keep the expected surplus to be equal at all posted prices, prices fall by less at high levels than at low levels. Thus, price dispersion increases, which motivates other visitors to increase search effort in order to find lower prices. The term $-L'(0)\pi$ discussed earlier captures this reward for a visitor's higher search effort. The countering force comes from the fact that prices are lower everywhere than before, which reduces visitors' incentive to search. The effect of price dispersion dominates if and only if prices are sufficiently dispersed. This occurs when search effort is higher than a threshold (3/2).[14]

The equilibrium is unique regardless of whether search effort is complements among visitors. The reason is that search effort is not strong complements, as shown by $S_1(s, z^*) < 1$. Two forces limit a visitor's response in search effort to other visitors' search effort. One is that the decrease in price levels everywhere reduces the need to search. The other is the equilibrium effect that higher search effort tends to reduce z. Since search effort and z both increase a visitor's meeting rate, they are substitutes to a visitor. To induce a stayer to enter the market, a visitor expects to concede part of the match surplus to the stayer. This expected concession increases in z, because a higher z implies a lower meeting rate

[14]This condition can be expressed as a restriction on the parameters. Since x is an increasing function of y, x is above the threshold 3/2 if y is above some threshold.

for a stayer. When it is optimal for a visitor to increase search effort, it is also optimal for the visitor to lower z so as to reduce the expected concession to the stayer in a meeting. For any given s, a lower z mitigates the increase in the meeting rate for a visitor, which limits the response of a visitor's search effort to other visitors' search effort.

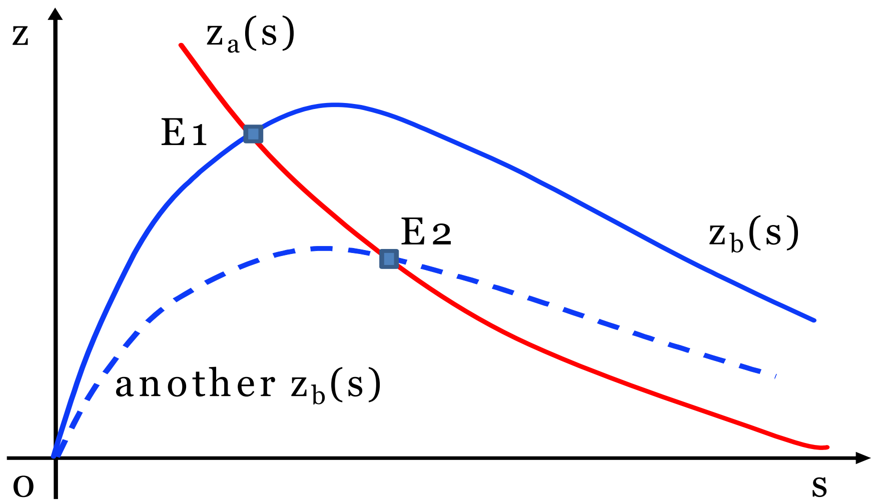

Figure 1. The unique equilibrium with endogenous search effort

Proposition 4.2 shows that a reduction in search cost has similar effects on price dispersion as an increase in the meeting efficiency studied by Corollary 3.4. Figure 1 illustrates the effects of a reduction in the marginal cost of search, ψ'. Before the reduction in search cost, let $z_b(s)$ be the solid curve in Figure 1. A reduction in search cost shifts the curve $z_b(s)$ to the dashed curve, which results in a higher s and a lower z. It is intuitive that a reduction is search cost induces a visitor to search with higher effort and leads to a higher meeting rate for a visitor. As the matching probability for a stayer falls, a smaller measure of stayers enter the market, resulting in a fall in z. Because x increases, dispersion widens in both posted and transaction prices. Under the mild condition $\varepsilon' \geq 0$, posted and transaction prices fall in the first-order stochastic dominance.[15]

[15] In a different model, Bethune et al. (2018) show that a reduction in the cost of becoming informed of prices can increase price dispersion.

4.2. Social inefficiency and corrective policies

The complementarity in search effort raises the question whether the SMSE is socially efficient. To address this question, I incorporate visitors' search effort into the social planner's problem in section 3.3. The total cost of stayers' entry is kN_s. Because $N_s = \theta(z)(s+s_0)N_v$, the entry cost divided by the measure of visitors is $k\theta(z)(s+s_0)$. Thus, social welfare normalized by the fixed measure of visitors is:

$$-\psi(s) + y\left(1-e^{-x}\right) - k\theta(z)(s+s_0).$$

Maximizing social welfare, the planner's choices (z,s) satisfies the first-order conditions:

$$\theta'(z) = \frac{y}{k}e^{-x}, \quad \psi'(s) = k\theta(z)\left[\varepsilon(z)-1\right]. \tag{4.10}$$

Compare these conditions with the counterparts in the SMSE, (4.9). The first condition is the same as in the SMSE. The second condition, which characterizes the socially efficient s, differs from that in the SMSE in the absence of the term $\frac{x}{2}$ on the right-hand side. This difference implies the following proposition (see Appendix A for a proof):

Proposition 4.3. *Relative to the social optimum, the SMSE has excessive search effort s, a deficient z, and an excessive meeting rate for a visitor, x.*

The inefficiency arises from an externality created by visitors' search effort. There is a private benefit for a visitor to increase search effort $\tilde{s}$ above other visitors' unexpectedly. Near $\tilde{s} = s$, this private benefit is: $-\pi x L'(0) = \pi x/2$. But if all visitors increase search effort, congestion increases, which is a negative externality. In the SMSE, search effort and the meeting rate for a visitor are inefficiently high. To maintain the optimal tradeoff between the meeting rate and the surplus of a trade, visitors choose an inefficiently low z. This low z mitigates, but does not correct completely, the excessive x.

Two elements are important for the inefficiency – the inability to commit to search effort and the presence of noisy matching. To see the importance, first suppose that visitors can commit to search effort before entering the market and so submarkets are indexed by

(p_H, z, s) instead of only (p_H, z). In this case, the optimal tradeoff between search effort and the trading probability will internalize the externality by "pricing" search effort correctly. Next, if search is purely directed, as in Burdett et al. (2001), endogenizing search effort does not generate inefficiency. This should be true more generally if search is directed by mechanisms instead of posted prices. The purpose for a visitor to increase search effort is to find a better mechanism. If there is no ex ante dispersion among mechanisms in the equilibrium, then endogenizing visitors' search effort should not lead to inefficiency even though there is ex post dispersion in transaction prices within a mechanism.

To restore efficiency and maintain a balanced budget, the government can consider the following policies: a subsidy rate σ_e to the entry cost, a proportional tax rate τ_y on the joint value of a trade, and a lump-sum rebate σ_n to a visitor's participation in the market.[16] With the subsidy σ_e, competitive entry of stayers into the submarket pushes a stayer's expected surplus conditional on having a meeting to:

$$\pi(z) = \frac{(1-\sigma_e) k}{\lambda(z)}. \tag{4.11}$$

With this modified formula of $\pi(z)$, the inverse of the distribution of posted prices is still given by (4.2). A visitor's expected surplus in the market is given by (4.4), with y being replaced by $(1-\tau_y) y$ and ψ by $\psi - \sigma_n$. This expected surplus is:

$$\begin{aligned} D(z, \tilde{s}) = \; & -\psi(\tilde{s}) + \sigma_n + [(1-\tau_y) y]\left[1 - e^{-\tilde{x}}\right] \\ & - (1-\sigma_e) k\theta(z)(\tilde{s} + s_0) L((\tilde{s} - s) z). \end{aligned}$$

In the symmetric SMSE, the first-order conditions of $(z, \tilde{s})$ become:

$$\begin{aligned} \theta'(z) &= \tfrac{(1-\tau_y)y}{(1-\sigma_e)k} e^{-x}, \\ \psi'(s) &= (1-\sigma_e) k\theta(z)\left[\varepsilon(z) - 1 + \tfrac{x}{2}\right]. \end{aligned} \tag{4.12}$$

Comparing these conditions with (4.10) for the social optimum, it is easy to verify that the two sets of conditions coincide if and only if

$$\tau_y = \sigma_e = \left[\frac{2[\varepsilon(z) - 1]}{x} + 1\right]^{-1}, \tag{4.13}$$

[16] A tax on visitors' search effort can also restore efficiency, but it requires the authority to have the unrealistic ability of observing visitors' search effort.

where (z, s) are the quantities in the social optimum. Because the measure of matches is $1 - e^{-x}$, total tax revenue on the joint output across matches is $y\tau_y [1 - e^{-x}]$. Total subsidy to the entry of stayers is $\sigma_e k (s + s_0) \theta$. The difference between the two is the lump-sum rebate to visitors that balances the government budget. That is,

$$\sigma_n = \sigma_e [(e^x - 1) \varepsilon - x] \frac{k\theta}{z}. \tag{4.14}$$

With the above $(\sigma_e, \tau_y, \sigma_n)$, it can be verified that D in the SMSE is equal to social welfare. The following proposition holds (see Appendix B for a proof):

Proposition 4.4. *Suppose that $(\sigma_e, \sigma_n, \tau_y)$ satisfy (4.13)-(4.14). (i) The policies induce the equilibrium to be socially efficient and maintain a balanced budget. (ii) The efficient policies increase z and reduce the following variables: s, x, aggregate output $Y = y [1 - e^{-x}]$, and (cv_F, cv_G). (iii) If $\varepsilon' (z) \geq 0$, the efficient policies reduce $\pi (z)$, reduce $(p_L, p_H, \Delta p)$, and reduce prices in the first-order stochastic dominance in $F (p|z)$ and $G (p|z)$.*

The policies (σ_e, τ_y) restore social efficiency of the equilibrium while the rebate σ_n balances the government budget. With the subsidy to a stayer's entry, σ_e, a stayer can break-even on entry with a smaller expected concession from a visitor in a meeting. A visitor's incentive to search weakens, which improves efficiency. However, σ_e distorts a visitor's tradeoff between the match surplus and the meeting rate by inducing a visitor to enter a submarket with an inefficiently high z. A tax on the joint output in a match, τ_y, eliminates this distortion by increasing the importance of the match surplus in a visitor's tradeoff relative to the meeting rate. The two policies (σ_e, τ_y) generate a budget surplus that is rebated to visitors through σ_n.

The corrective policies also affect price levels and price dispersion. In the absence of the policies, search effort is excessive and price dispersion is inefficiently wide. The policies tame search effort by reducing price dispersion. Price levels respond to the policies ambiguously in general. However, if $\varepsilon' \geq 0$, then posted and transaction prices fall in the first-order stochastic dominance. The derivative ε' measures the sensitivity of θ', as discussed for Proposition 3.4, and the condition $\varepsilon' \geq 0$ is satisfied by well-known meeting functions (see Example 2.2). When $\varepsilon' \geq 0$, the responses of z and $\lambda(z)$ to the policies are relatively weak. In this case, the policies affect a stayer's expected surplus conditional on having a meeting, $\pi(z)$, primarily through the effective cost of entry, $(1-\sigma_e)k$. Because σ_e reduces this effective cost, it reduces the level of π needed for a stayer to break-even on entry. The fall in π is delivered by lower prices.

4.3. Management of aggregate activities

I focus on how the corrective policies depend on y. A higher y is a better economic condition, arising from aggregate supply or demand. In Figure 1, an increase in y shifts up the curve $z_a(s)$ and leaves the curve $z_b(s)$ intact. Equilibrium s increases. Equilibrium z increases if the SMSE is one like point E1, but decreases if the SMSE is one like point E2. The following proposition states how the SMSE and the corrective policies respond to y (see Supplementary Appendix D for a proof):

Proposition 4.5. *Without the corrective policies, an increase in* y *has the following effects on the SMSE:* $\frac{ds}{dy} > 0$, $\frac{dx}{dy} > 0$, $\frac{dp_H}{dy} > 0$, $\frac{d\Delta p}{dy} > 0$, $\frac{d}{dy}cv_F > 0$, *and* $\frac{d}{dy}cv_G > 0$. *If* $\psi''' > 0$, *then* $\frac{dz}{dy} > 0$. *If* $\psi''' < 0$, *then there exists* $s_a \in (0, \infty]$ *such that* $\frac{dz}{dy} > 0$ *iff* $s < s_a$. *With the efficient policies in (4.13), a sufficient condition for* $\frac{d\sigma_e}{dy} > 0$ *and* $\frac{d\tau_y}{dy} > 0$ *is* $\varepsilon'(z) \leq (\varepsilon - 1)/z$.

An increase in y increases the market tightness, increases the meeting rate for a visitor, increases the two bounds on prices and shifts the price distributions to higher prices. These effects are intuitive. When the joint value of a trade increases, more stayers enter the market, which increases the market tightness. Equilibrium prices increase to be consistent with the optimal choices of both sides of the market. For visitors, the benefit of a higher y induces them to enter a submarket that has a higher meeting rate, which necessarily comes with higher prices. For stayers, the meeting probability falls because of the higher entry of stayers. However, the expected surplus for a stayer in the market (unconditional on a meeting) must remain the same as the fixed cost of entry. This implies that a stayer's expected surplus conditional on a meeting must rise. For this to happen despite the decrease in the stayer's trading probability, prices must increase to raise the stayer's surplus conditional on a trade. Moreover, as explained for Corollary 3.1, the increase in a visitor's meeting rate x widens price dispersion by reducing the trading probability for a stayer by more at high prices than at low prices.[17]

Without the corrective policies, an improvement in the economic condition results in an over-heated economy. Search effort and a visitor's meeting rate increase by an excessive amount, resulting in inflated prices and inefficiently wide dispersion in prices. The policies described by Proposition 4.4 are automatic stabilizers for the economy even if the policies do not respond to the increase in y. These policies tame visitors' search effort, moderate the increase in aggregate output and prevent price dispersion from widening excessively. Moreover, Proposition 4.5 shows that if $\varepsilon' \leq (\varepsilon - 1)/z$, the tax on the joint value of a trade should increase in an economic boom and decrease in a recession. To accompany this procyclical tax, the subsidy to stayers' entry should also be procyclical.

The corrective policies contrast with those in Diamond (1982), who assumes the matching function to have increasing returns to scale. In the current model, the meeting and the matching functions have constant returns to scale. Instead, the inefficiency arises from an externality in visitors' choice of search effort. Also, in contrast to Diamond's recommendation to manage aggregate demand, the corrective policies in Proposition 4.5 call for a policy mix that manages both aggregate demand and supply.

5. Conclusion

This paper constructs a tractable model of sequentially mixed search, which has directed search followed by noisy matching with multiple offers. I establish existence of a unique equilibrium and analyze the novel implications of the equilibrium on quantities and price dispersion. Moreover, I show that an increase in the meeting efficiency widens price dis-

[17] The variable z responds to a higher y ambiguously. Because a higher y induces search effort to increase, the surplus that a visitor expects to concede to a stayer in a meeting increases. To restrict the increase in this concession, a visitor may reduce z. This happens when search effort is high. When search effort is low, the direct effect of y on z dominates, in which case z increases.

persion. An extension endogenizing search effort shows that the equilibrium is constrained inefficient, where search effort is inefficiently high and can be strategic complements among visitors. Under a mild condition, policies that restore efficiency should lean against the wind to manage aggregate demand and supply.

The simple framework of SMS in this paper allows for several extensions. An obvious extension is to introduce heterogeneity among visitors, say, in their valuation of the good. This valuation can be either public or private information. In both cases, the modifications to the SMSE are straightforward. Visitors of different types will self-select into different submarkets indexed by (p_H, x), with higher-valuation visitors selecting a higher x and a higher p_H. As in this paper, the SMSE is constrained efficient when search effort is exogenous and inefficient when search effort is endogenous.

A more substantial extension is to allow for many-to-many meetings. That is, each stayer and each visitor can both meet more than one individual on the other side of the market. This extension needs to specify a procedure by which individuals select trading partners from the meetings. In a sequel (Shi, 2018), I have studied many-to-many meetings with two selection mechanisms. In one, stayers select visitors, where stayers posting lower prices select earlier. In the other, visitors selected stayers according to a predetermined order of visitors. Both selection mechanisms lead to the same expected payoffs to the market participants and constrained efficiency of the equilibrium.

The third extension is to incorporate trading relationships and business cycles. In the goods market, past trades create customer relationships, as analyzed in Gourio and Rudanko (2014) and Shi (2016). In the labor market, employment is a lasting relationship between a firm and a worker, as analyzed by Burdett and Mortensen (1998). In both markets, individuals in a relationship can continue to search. By embedding search on the relationship into a business cycle model, the analysis can shed light on how the turnover of relationships and the price distribution fluctuate in the business cycle. Directed search in the first stage of SMS will be important for such an analysis to be tractable by making the SMSE block recursive, as formulated by Shi (2009) and Menzio and Shi (2010).

Appendix

A. Proofs for Propositions 4.2 and 4.3

Proof of Proposition 4.2:

First, I prove that a visitor's optimal choices $(z, \tilde{s})$ are interior and satisfy (4.7) and (4.8). Because $D(z, \tilde{s})$ is differentiable in $(z, \tilde{s})$, the optimal choices satisfy the first-order conditions if they are interior. The derivative $D_1(z, \tilde{s})$ is equal to the right-hand side of (4.7) multiplied by $(\tilde{s} + s_0)$. Since $\theta(0) = 0$, $L(0) = 1$ and $L'(0) = -\frac{1}{2}$, then

$$D_1(0, \tilde{s}) = (\tilde{s} + s_0)[y - k\theta'(0)] > 0 \text{ for all } \tilde{s} \geq 0,$$

where the inequality is a maintained assumption. Similarly, $D_1(\infty, \tilde{s}) < 0$. Thus, for all $\tilde{s} > 0$, the optimal z is interior and satisfies the first-order condition (4.7). $D_2(z, \tilde{s})$ is given by the right-hand side of (4.8). In any symmetric SMSE, I have:

$$D_2(z, s) = -\psi' + yze^{-x} - k\theta\left[1 - \frac{x}{2}\right].$$

Substituting ye^{-x} from the version of (4.7) in the symmetric SMSE, I get:

$$D_2(z, s) = -\psi' + k\theta\left[\varepsilon - 1 + \frac{x}{2}\right].$$

Since $\psi'(0) = 0$, then $D_2(z, 0) > 0$ for $z > 0$. That is, in any symmetric SMSE, $\tilde{s} = s > 0$. Also, $D_2(z, \infty) < 0$, and so the optimal $\tilde{s}$ satisfies $\tilde{s} < \infty$. Thus, in any symmetric SMSE, the optimal $\tilde{s}$ is interior and satisfies the first-order condition (4.8).

Second, I prove that a unique SMSE exists. Recall that, given any s, $z_a(s)$ denotes the solution for z to the first equation in (4.9) and $z_b(s)$ the solution to the second equation. The equilibrium value of s solves $z_b(s) - z_a(s) = 0$. The assumption $y > k\theta'(0)$ implies $z_a(0) > 0$. Also, $z_a(\infty) = 0$. Since $\psi'(0) = 0$ and $\theta(0) = 0$, then $z_b(0) = 0 < z_a(0)$ and $z_b(\infty) > 0 = z_a(\infty)$. Thus, the equation $z_b(s) = z_a(s)$ has at least one solution in $(0, \infty)$. To prove that the solution is unique, compute:

$$z_a'(s) = \frac{-z}{\frac{\theta''}{\theta'} + s + s_0} < 0, \quad z_b'(s) = \frac{\psi'' - kz\theta/2}{k\left[(\varepsilon + 1)\frac{s+s_0}{2} + z\theta''\right]}.$$

Here I substituted $\varepsilon' = \frac{z\theta''}{\theta} - \frac{\varepsilon(\varepsilon-1)}{z}$. The difference $[z_b'(s_0) - z_a'(s_0)]$ has the same sign as

$$\left(\frac{\theta''}{\theta'} + s + s_0\right)\psi'' + k\theta\left[\frac{z\theta''}{\theta'}\left(\varepsilon - \frac{1}{2}\right) + \varepsilon\frac{x}{2}\right].$$

Since this is positive, then $z_b'(s) > z_a'(s)$ for all $s \geq 0$. Thus, the solution to $z_a(s) = z_b(s)$ is unique, as multiple solutions necessarily have alternating signs of $[z_b'(s) - z_a'(s)]$.

Third, I prove $S_1(s, z^*) < 1$ and find the condition for $S_1(s, z^*) > 0$, where $S(s, z)$ is a visitor's optimal choice of $\tilde{s}$ given (s, z). Differentiating (4.8), I can compute:

$$S_1(s, z) = \frac{k\theta(z)z}{-D_{22}(z, \tilde{s})}[L' + \tilde{x}L''].$$

In the symmetric SMSE, $\tilde{s} = s$. Using the features of L in Lemma 4.1, I can evaluate:

$$\begin{aligned} D_{22}(z^*, s) &= -\psi'' - kz\theta[\varepsilon - 1 + x/3] < 0, \\ S_1(s, z^*) &= \tfrac{k\theta(z^*)z^*}{-D_{22}(z^*, s)}\left[\tfrac{x^*}{3} - \tfrac{1}{2}\right]. \end{aligned}$$

Here z^* is used instead of z because the first-order condition of z, (4.7), is used to substitute the term $ye^{-\tilde{x}}$. Thus, $S_1(s, z^*) > 0$ if and only if $x^* > \frac{3}{2}$. Moreover, $S_1(s, z^*) < 1$ if and only if $\psi'' + kz\theta\left(\varepsilon - \frac{1}{2}\right) > 0$, which is satisfied.

Finally, suppose $\psi(s) = \psi_0\hat{\psi}(s)$ for some constant $\psi_0 > 0$ and a function $\hat{\psi}(s)$. An increase in ψ_0 is an exogenous increase in the marginal cost of search effort. A reduction in ψ_0 is an exogenous reduction in the marginal cost of search effort. Differentiating (4.9) with respect to ψ_0 shows $\frac{\mathrm{d}z}{\mathrm{d}\psi_0} > 0$, $\frac{\mathrm{d}s}{\mathrm{d}\psi_0} < 0$, and $\frac{\mathrm{d}x}{\mathrm{d}\psi_0} < 0$. Since $cv_F'(x) > 0$ and $cv_G'(x) > 0$ by Corollary 3.1, then $\frac{\mathrm{d}}{\mathrm{d}\psi_0}cv_F < 0$ and $\frac{\mathrm{d}}{\mathrm{d}\psi_0}cv_G < 0$. Similar to the proof of the effects of σ_e later in Proposition 4.4, the increase in z and the fall in x imply that, if $\varepsilon' \geq 0$, then $\frac{\mathrm{d}F}{\mathrm{d}\psi_0} < 0$ and $\frac{\mathrm{d}G}{\mathrm{d}\psi_0} < 0$. That is, an in ψ_0 increase prices in the first-order stochastic dominance in F and G. **QED**

Proof of Proposition 4.3:

The first equation in (4.10) is identical to the first equation in (4.9). For any given s, the common solution for z to the two equations is $z_a(s)$. Let $z_b^e(s)$ denote the solution for the equilibrium z to the second equation in (4.9), and $z_b^o(s)$ the solution for the socially optimal z to the second equation in (4.10). Denote $\Delta z^e(s) = z_b^e(s) - z_a(s)$ and $\Delta z^o(s) =$

$z_b^o(s) - z_a(s)$. Then, the equilibrium s solves $\Delta z^e(s) = 0$, and the socially optimal s solves $\Delta z^o(s) = 0$. As shown in the proof of Proposition 4.2, $\Delta z^{e\prime}(s) > 0$. Similarly, $\Delta z^{o\prime}(s) > 0$. For any $s \geq 0$ and $z > 0$, the right-hand side of the second condition in (4.9) is strictly greater than that of the second condition in (4.10). For any given $s \geq 0$, a smaller z is required to satisfy the second equation in (4.9) than to satisfy the second equation in (4.10). That is, $z_b^e(s) < z_b^o(s)$ for all $s \geq 0$. This implies $\Delta z^e(s) < \Delta z^o(s)$ for all $s \geq 0$. Since $\Delta z^{e\prime}(s) > 0$ and $\Delta z^{o\prime}(s) > 0$, the solution for s to $\Delta z^e(s) = 0$ must be strictly larger than the solution for s to $\Delta z^o(s) = 0$. Because z satisfies the common equation $z = z_a(s)$ in the SMSE and the social optimum, and because $z_a'(s) < 0$, then z is lower in the equilibrium z than in the social optimum. Backing out x from the common equation for $z_a(s)$ in the SMSE and in the social optimum, I conclude that x is higher in the SMSE than in the social optimum. This completes the proof of Proposition 4.3. **QED**

B. Proof of Proposition 4.4

(i) The text preceding the proposition has established the result that the policies $(\sigma_e, \tau_y, \sigma_n)$ induce the SMSE to be socially efficient.

(ii) By Proposition 4.3, the SMSE without the policies has an excessive s, a deficient z and an excessive x, relative to the social optimum. Because the efficient policies restore the social optimum, they reduce s, increase z and reduce x.

(iii) I analyze the effects of the policies on prices in three steps. First, I reduce the dimension of the policies and variables. To do so, let τ_y depend on σ_e as in (4.13). Then, for all σ_e, the first equation in (4.12) is the same as the first equation in (4.9), and the common solution for z to the two equations is $z = z_a(s)$. For any given s, $z_a(s)$ does not depend on the policies directly. Also, $z_a'(s) < 0$, as shown in the proof of Proposition 4.2. Substituting $z = z_a(s)$ into the second equation of (4.12), I get:

$$\psi'(s) = (1 - \sigma_e) k\theta(z) \left[\varepsilon(z) - 1 + \frac{x}{2}\right]_{z=z_a(s)}. \tag{B.1}$$

Second, I compute the effects of σ_e on (s, z), Y and (cv_F, cv_G), taking into account the dependence of τ_y on σ_e in (4.13). In this computation, σ_e is arbitrary instead of the efficient one. However, if the derivative of a variable with respect to σ_e is positive, then the variable increases under the efficient policies, because the efficient policies have $\sigma_e > 0$. Differentiating (B.1) with respect to σ_e, I get:

$$\frac{\mathrm{d}s}{\mathrm{d}\sigma_e} = \frac{-\psi'}{1-\sigma_e}\left[\psi'' + \frac{(2\varepsilon-1)\,z\theta'' + x\theta'\varepsilon}{\theta'' + (s+s_0)\,\theta'}\frac{1-\sigma_e}{2}k\theta\right]^{-1} < 0.$$

This implies $\frac{\mathrm{d}z}{\mathrm{d}\sigma_e} = z_a'(s)\frac{\mathrm{d}s}{\mathrm{d}\sigma_e} > 0$ and

$$\frac{\mathrm{d}x}{\mathrm{d}\sigma_e} = \frac{z\theta''}{\theta'' + (s+s_0)\,\theta'}\frac{\mathrm{d}s}{\mathrm{d}\sigma_e} < 0.$$

Since aggregate output is $Y = y\left[1-e^{-x}\right]$, it is clear that $\frac{\mathrm{d}Y}{\mathrm{d}\sigma_e} < 0$. The coefficients of variation in prices, $cv_F(x)$ and $cv_G(x)$, are given by (3.1). Since $cv_F'(x) > 0$ and $cv_G'(x) > 0$ by Corollary 3.1, and since $\frac{\mathrm{d}x}{\mathrm{d}\sigma_e} < 0$, then $\frac{\mathrm{d}}{\mathrm{d}\sigma_e}cv_F < 0$ and $\frac{\mathrm{d}}{\mathrm{d}\sigma_e}cv_G < 0$.

Third, I compute the effects of σ_e on $\pi(z)$ and prices, again taking into account the dependence of τ_y on σ_e. With the policies, $\pi(z) = \frac{(1-\sigma_e)k}{\lambda(z)}$. Using the above results for $\frac{\mathrm{d}s}{\mathrm{d}\sigma_e}$ and $\frac{\mathrm{d}z}{\mathrm{d}\sigma_e}$, I compute:

$$\begin{aligned}&-\tfrac{\lambda}{k}\left\{\left[\theta''+(s+s_0)\,\theta'\right]\psi'' + \left[(2\varepsilon-1)\,z\theta''+x\theta'\varepsilon\right]\tfrac{1-\sigma_e}{2}k\theta\right\}\tfrac{\mathrm{d}\pi(z)}{\mathrm{d}\sigma_e}\\ &= \left[(2\varepsilon-1)\,z\theta''+x\theta'\varepsilon\right]\tfrac{1-\sigma_e}{2}k\theta - (\varepsilon-1)\,\theta'\psi' + \left[\theta''+(s+s_0)\,\theta'\right]\psi''.\end{aligned}$$

For the right-hand side to be strictly positive, a sufficient condition is that the difference between the first two terms is non-negative. Substituting ψ' from (B.1) and substituting $z\theta'' = \theta\varepsilon' + (\varepsilon-1)\,\theta'$, I rewrite this difference as:

$$\frac{1-\sigma_e}{2}k\theta\left\{(2\varepsilon-1)\,\theta\varepsilon' + \left[\varepsilon-1+x\right]\theta'\right\}.$$

Since $\varepsilon > 1$, a sufficient condition for the above expression to be strictly positive is $\varepsilon' \geq 0$. Thus, $\varepsilon' \geq 0$ is sufficient for $\frac{\mathrm{d}\pi(z)}{\mathrm{d}\sigma_e} < 0$.

Assume $\varepsilon' \geq 0$. For any given p, (4.1) implies:

$$x\frac{\mathrm{d}F(p|z)}{\mathrm{d}\sigma_e} = -F\frac{\mathrm{d}x}{\mathrm{d}\sigma_e} - \frac{\mathrm{d}\pi(z)}{\mathrm{d}\sigma_e} > 0.$$

Thus, the policies reduce posted prices in the first-order stochastic dominance. Similar to (2.5), the distribution of transactions prices is:

$$G(p|z) \equiv \frac{1-e^{-xF(p|z)}}{1-e^{-x}} = \frac{1-\frac{\pi(z)}{p}}{1-e^{-x}},$$

where the second equality comes from substituting F from (4.1). Because σ_e reduces $\pi(z)$ and x, then $\frac{\mathrm{d}G(p|z)}{\mathrm{d}\sigma_e} > 0$ for any given p. That is, the policies reduce transaction prices in the first-order stochastic dominance. Moreover, (4.2) implies:

$$\frac{\mathrm{d}p_L}{\mathrm{d}\sigma_e} = \frac{\mathrm{d}\pi(z)}{\mathrm{d}\sigma_e} < 0, \; \frac{\mathrm{d}p_H}{\mathrm{d}\sigma_e} = e^{xF}\left[\frac{\mathrm{d}\pi(z)}{\mathrm{d}\sigma_e} + \pi(z) F \frac{\mathrm{d}x}{\mathrm{d}\sigma_e}\right] < 0.$$

The policies affect the price spread, $\Delta p = p_H - p_L$, as follows:

$$\frac{\mathrm{d}\Delta p}{\mathrm{d}\sigma_e} = \left(e^{xF} - 1\right)\frac{\mathrm{d}\pi(z)}{\mathrm{d}\sigma_e} + \pi(z) F e^{xF} \frac{\mathrm{d}x}{\mathrm{d}\sigma_e} < 0.$$

This completes the proof of Proposition 4.4. **QED**

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Supplementary Appendix for

"Sequentially Mixed Search and Equilibrium Price Dispersion"

Shouyong Shi

C. Proofs of Proposition 3.4 and Lemma 4.1

Proof of Proposition 3.4:

I prove the following proposition that includes the effects of y, in addition to the effects of A stated in Proposition 3.4:

Proposition C.1. *(i)* $\frac{dx^*}{dy} > 0$, $\frac{d\theta}{dy} > 0$, $\frac{dp_L}{dy} > 0$, $\frac{dp_H}{dy} > 0$, $\frac{d\Delta p}{dy} > 0$, *and prices increase in* y *in the first-order stochastic dominance in* F *and* G. *(ii)* $\frac{dx^*}{dA} > 0$, $\frac{d}{dy}cv_i(x) > 0$ *and* $\frac{d}{dA}cv_i(x) > 0$ *for* $i = F, G$. *(iii)* $\frac{d\theta}{dA} > 0$ *iff* $A < A_0$, *where* A_0 *is defined by (3.3). (iv)* $\frac{dp_H}{dA} \leq 0$ *iff* $(A_0 - A)\varepsilon' > 0$*; If* $\varepsilon' \geq 0$ *then* $\frac{dp_L}{dA} < 0$ *and* $\frac{d\Delta p}{dA} > 0$. *(v) If* $\varepsilon' \geq 0$ *and* $A < A_0$, *an increase in* A *reduces posted prices and transaction prices in the first-order stochastic dominance.*

Proof. Taking ln on (3.2) and differentiating yields:

$$\left(\frac{\nu\theta''}{\theta'} + x\right)\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{y} + \left(\frac{\nu\theta''}{\theta'} + 1\right)\frac{\mathrm{d}A}{A}, \tag{C.1}$$

where the asterisk on x is suppressed and the argument of θ is $\nu = x/A$. Since $\theta' > 0$ and $\theta'' > 0$, then $\frac{\mathrm{d}x}{\mathrm{d}y} > 0$ and $\frac{\mathrm{d}x}{\mathrm{d}A} > 0$. The coefficients of variation given in (3.1) are only functions of x. Moreover, $cv_F'(x) > 0$ and $cv_G'(x) > 0$ (see Corollary 3.1). Because $\frac{\mathrm{d}x}{\mathrm{d}y} > 0$ and $\frac{\mathrm{d}x}{\mathrm{d}A} > 0$, then $\frac{\mathrm{d}}{\mathrm{d}y}cv_i(x) > 0$ and $\frac{\mathrm{d}}{\mathrm{d}A}cv_i(x) > 0$ for $i = F, G$.

Differentiating $\theta(\nu)$ and substituting $(\mathrm{d}x)$ yields:

$$\left(\frac{\nu\theta''}{\theta'} + x\right)\frac{\mathrm{d}\theta}{\nu\theta'} = \frac{\mathrm{d}y}{y} + (1-x)\frac{\mathrm{d}A}{A}.$$

Clearly, $\frac{\mathrm{d}\theta}{\mathrm{d}y} > 0$. The definition of A_0 in (3.3) implies that $x < 1$ is equivalent to $A < A_0$. Then, $\frac{\mathrm{d}\theta}{\mathrm{d}A} > 0$ iff $x < 1$ and, hence, iff $A < A_0$. Substituting π from (2.9) into (2.8) and differentiating, I get:

$$\frac{x}{k\theta}\left(\frac{\nu\theta''}{\theta'}+x\right)\mathrm{d}p_L=\frac{\varepsilon-1}{y}\mathrm{d}y-\left(\frac{\theta}{\theta'}\varepsilon'+x\varepsilon\right)\frac{\mathrm{d}A}{A} \tag{C.2}$$

$$\frac{xe^{-x}}{k\theta}\left(\frac{\nu\theta''}{\theta'}+x\right)\mathrm{d}p_H=\frac{\varepsilon-1+x}{y}\mathrm{d}y-\frac{\theta\varepsilon'}{\theta'}(1-x)\frac{\mathrm{d}A}{A} \tag{C.3}$$

$$\begin{aligned}\frac{xe^{-x}}{k\theta}\left(\frac{\nu\theta''}{\theta'}+x\right)\mathrm{d}\Delta p= & \frac{\left(1-e^{-x}\right)(\varepsilon-1)+x}{y}\mathrm{d}y \\ & +\left[\left(x-1+e^{-x}\right)\frac{\theta}{\theta'}\varepsilon'+x\varepsilon e^{-x}\right]\frac{\mathrm{d}A}{A}\end{aligned} \tag{C.4}$$

I have used $\varepsilon'(\nu)=\frac{1}{\theta}\left[\nu\theta''-(\varepsilon-1)\theta'\right]$, where the argument of θ and ε is $\nu=x/A$. Because $\varepsilon>1$, the above equations show that $\frac{\mathrm{d}p_L}{\mathrm{d}y}>0$, $\frac{\mathrm{d}p_H}{\mathrm{d}y}>0$ and $\frac{\mathrm{d}\Delta p}{\mathrm{d}y}>0$. The effects of A on $(p_L,p_H,\Delta p)$ depend on ε'. Clearly, $\frac{\mathrm{d}p_H}{\mathrm{d}A}\leq 0$ iff $(1-x)\varepsilon'>0$ and, hence, iff $(A_0-A)\varepsilon'>0$. Note that $x-1+e^{-x}>0$ for all $x>0$. If $\varepsilon'\geq 0$, then $\frac{\mathrm{d}p_L}{\mathrm{d}A}<0$ and $\frac{\mathrm{d}\Delta p}{\mathrm{d}A}>0$.

The cumulative distribution function of posted prices is $F(p|x)$ given by (2.6) and the cumulative distribution function of transaction prices is $G(p|x)$ given by (2.5). For any given p, differentiating these functions with respect to (y,A) yields:

$$x\left(\frac{\nu\theta''}{\theta'}+x\right)\mathrm{d}F=-\frac{\varepsilon-1+xF}{y}\mathrm{d}y+\left[(1-xF)\frac{\theta}{\theta'}\varepsilon'+x\varepsilon(1-F)\right]\frac{\mathrm{d}A}{A} \tag{C.5}$$

$$\begin{aligned}x\left(\frac{\nu\theta''}{\theta'}+x\right)\frac{\mathrm{d}G}{G}= & -\left(\frac{\varepsilon-1}{e^{xF}-1}+\frac{x}{e^{x}-1}\right)\frac{\mathrm{d}y}{y} \\ & +\left[\frac{\theta\varepsilon'}{\theta'}\left(\frac{1}{e^{xF}-1}-\frac{x}{e^{x}-1}\right)+x\varepsilon\left(\frac{1}{e^{xF}-1}-\frac{1}{e^{x}-1}\right)\right]\frac{\mathrm{d}A}{A}.\end{aligned} \tag{C.6}$$

Because $\varepsilon>1$, the coefficients of $(\mathrm{d}y)$ in both equations are strictly negative. Thus, an increase in y increases both posted and transaction prices in the first-order stochastic dominance. Note that $xF\leq x$. If $\varepsilon'\geq 0$ and $A<A_0$ (i.e., $x<1$), then the coefficients of $(\mathrm{d}A)$ in (C.5) and (C.6) are strictly positive for all interior p. Thus, an increase in A reduces both posted and transaction prices. **QED**

Proof of Lemma 4.1:

Using L'Hopital's rule, one can verify the properties in (i) of Lemma 4.1. To verify $L(t)>0$ in (ii) of the Lemma, note that $1>e^{-t}$ if and only if $t>0$. Thus, $L(t)>0$ for all $t \neq 0$. In addition, $L(0) = 1 > 0$, as proven in part (i). To prove $L'(t) < 0$ for all t, it suffices to prove $L'(t) < 0$ for all $t \neq 0$, since $L'(0) = -\frac{1}{2} < 0$ by (i). Compute

$$L'(t) = \frac{1}{t^2}\left[(t+1)e^{-t} - 1\right].$$

Examine the expression in [.] on the right-hand side for $t \neq 0$. The derivative of the expression with respect to t is equal to $-te^{-t}$, which is positive if and only if $t < 0$. Thus, the expression is maximized as $t \to 0$. Because the expression approaches 0 as $t \to 0$, the expression is negative for all $t \neq 0$. Thus, $L'(t) < 0$ for all $t \neq 0$. Similarly, compute:

$$L''(t) = \frac{1}{t^3}\left[2 - \left(t^2 + 2t + 2\right)e^{-t}\right].$$

The expression in [.] on the right-hand side is increasing in t and, at $t = 0$, it is equal to 0. Thus, the expression is positive if and only if $t > 0$. Since $t^3 > 0$ if and only if $t > 0$, then $L''(t) > 0$ for all t. **QED**

D. Proof of Proposition 4.5

Comparative statics with respect to y without the policies: In the absence of the policies, the SMSE is characterized by (4.9). For any given s, the solution for z is $z_a(s)$ to the first equation in (4.9) and $z_b(s)$ to the second equation. To prove $\frac{\mathrm{d}s}{\mathrm{d}y} > 0$, note that $\frac{\partial z_a(s)}{\partial y} > 0$ and $\frac{\partial z_b(s)}{\partial y} = 0$, where the partial derivatives are taken for any given s. Since $z_b'(s) > z_a'(s)$ with the optimal s, then

$$\frac{\mathrm{d}s}{\mathrm{d}y} = \frac{\partial z_a(s)/\partial y}{z_b'(s) - z_a'(s)} > 0.$$

Because $\frac{\partial z_b(s)}{\partial y} = 0$, then

$$\frac{\mathrm{d}z}{\mathrm{d}y} = z_b'(s)\frac{\mathrm{d}s}{\mathrm{d}y}.$$

Therefore, the sign of $\frac{\mathrm{d}z}{\mathrm{d}y}$ is the same as $z_b'(s)$, which is ambiguous and examined below.

Similarly, using the notation $x = (s + s_0)z$ to express $s = \frac{x}{z} - s_0$, I can use the two equations in (4.9) to solve for $z = z_a(x)$ and $z = z_b(x)$. Then,

$$z_a'(x) = \frac{-\theta'}{\theta''}, \quad z_b{}'(x) = \frac{\psi'' - \frac{k\theta}{2}z}{kz\left[z\theta'' + \frac{x}{2}\theta'\right] + \psi''\frac{x}{z}}.$$

Again, it can be verified that $z_b'(x) > z_a'(x)$. Thus, there is a unique solution x to $z_b(x) = z_a(x)$, and the solution satisfies $\frac{\mathrm{d}x}{\mathrm{d}y} > 0$. Furthermore, $\frac{\mathrm{d}(\pi e^x)}{\mathrm{d}y} > 0$, and so $\frac{\mathrm{d}p_H}{\mathrm{d}y} > 0$. If $\frac{\mathrm{d}z}{\mathrm{d}y} \leq 0$, then $\frac{\mathrm{d}p_L}{\mathrm{d}y} \leq 0$ and $\frac{\mathrm{d}\Delta p}{\mathrm{d}y} > 0$. If $\frac{\mathrm{d}z}{\mathrm{d}y} > 0$, then $\frac{\mathrm{d}\pi}{\mathrm{d}y} > 0$ and

$$\frac{\mathrm{d}\Delta p}{\mathrm{d}y} = (e^x - 1)\frac{\mathrm{d}\pi}{\mathrm{d}y} + \pi e^x \frac{\mathrm{d}x}{\mathrm{d}y} > 0.$$

In all cases, $\frac{\mathrm{d}\Delta p}{\mathrm{d}y} > 0$. Since $cv_F'(x) > 0$ and $cv_G'(x) > 0$, the result $\frac{\mathrm{d}x}{\mathrm{d}y} > 0$ implies $\frac{\mathrm{d}}{\mathrm{d}y}cv_F > 0$ and $\frac{\mathrm{d}}{\mathrm{d}y}cv_G > 0$.

Finally, I establish the sign of $z_b'(s)$. Return to the use of s instead of x as the variable. Denote $z_c(s)$ as the solution to $\theta(z_c)z_c = 2\psi''(s)/k$. Then, $z_b'(s) > 0$ if and only if $z < z_c(s)$. Because $\psi'' > 0$, then $z_c(s_0) > 0$. Consider the following cases:

Case (i): $\psi''' > 0$. In this case, $z_c'(s) > 0$ for all s. I prove that $z_b'(s) > 0$ for all $s \geq s_0$, which implies $\frac{\mathrm{d}z}{\mathrm{d}y} > 0$ by the above proof. A sufficient condition for this result is $z_b(s) < z_c(s)$ for all $s \geq 0$. To prove that this sufficient condition holds, suppose, to the contrary, that $z_b(s_a) = z_c(s_a)$ for some $s_a \in [0, \infty)$. Clearly, $s_a > 0$ and $z_b'(s_a) = 0$. Without loss of generality, let s_a be the smallest solution to $z_b(s) = z_c(s)$. Because $z_b(s) < z_c(s)$ for all $s < s_a$, and $z_b(s_a) = z_c(s_a)$, then $z_b'(s_a) \geq z_c'(s_a) > 0$. This contradicts the fact that $z_b'(s_a) = 0$. Thus, $s_a > 0$ does not exist, and so $z_b(s) < z_c(s)$ for all $s > s_0$. In this case, the SMSE is depicted by point E1 in Figure 1.

Case (ii): $\psi''' < 0$, but $z_b(s) < z_c(s)$ for all s. As in case (i), this case has $z_b'(s) > 0$ for all s, and so $\frac{\mathrm{d}z}{\mathrm{d}y} > 0$.

Case (iii): $\psi''' < 0$, and there exists $s_a > 0$ such that $z_b(s_a) = z_c(s_a)$. Let s_a be the smallest solution to $z_b(s) = z_c(s)$. I prove that s_a is the only solution to $z_b(s) = z_c(s)$. Suppose, to the contrary, that there is another solution s_1 ($> s_a$) to $z_b(s) = z_c(s)$. Without loss of generality, let s_1 be the smallest solution among all $s > s_a$. Then $z_b(s_1 - \varepsilon) > z_c(s_1 - \varepsilon)$ for sufficiently small $\varepsilon > 0$. This fact and the definition of s_1 imply $z_b'(s_1) \leq z_c'(s_1) < 0$. This contradicts the fact that $z_b'(s_1) = 0$. Thus, s_1 does not exist; i.e., $z_b(s) > z_c(s)$ for all $s > s_a$. Therefore, $z_b'(s) < 0$ if and only if $s > s_a$. That is, $\frac{\mathrm{d}z}{\mathrm{d}y} > 0$ if and only if the optimal s satisfies $s < s_a$. In this case, the SMSE is depicted by point E2 in Figure 1.

Case (iv): $\psi''' = 0$. In this case, $z_c(s)$ is constant over s. If $z_b(s) < z_c(s)$ for all $s > 0$, the case is qualitatively the same as case (i). If there is a solution $s_a > 0$ to $z_b(s) = z_c(s)$, then $z_b(s) = z_c(s)$ for all $s > s_a$. In this case, $\frac{\mathrm{d}z}{\mathrm{d}y} > 0$ if $s < s_a$, and $\frac{\mathrm{d}z}{\mathrm{d}y} = 0$ if $s > s_a$.

The effect of y on the efficient policies: Under the efficient policies in (4.13), the equilibrium allocation coincides with the social optimum given by (4.10). Differentiating (4.10) with respect to y, I get:

$$\frac{\mathrm{d}z}{\mathrm{d}y} = \frac{e^{-x}}{k\left[(s+s_0)\,\theta' + \left(\frac{z^2 k\theta'}{\psi''} + 1\right)\theta''\right]} > 0, \qquad \frac{\mathrm{d}s}{\mathrm{d}y} = \frac{kz\theta''}{\psi''}\frac{\mathrm{d}z}{\mathrm{d}y} > 0.$$

I have substituted $\varepsilon' = \frac{1}{\theta}\left[z\theta'' - (\varepsilon - 1)\,\theta'\right]$. All variables in the above expressions, as in the remainder of this proof, are the ones in the social optimum. Differentiating the expression for σ_e in (4.13) with respect to y, I have:

$$\frac{\mathrm{d}\sigma_e}{\mathrm{d}y} = \left(\frac{2\mathrm{d}z}{\mathrm{d}y}\right)\frac{\sigma_e^2}{x}\left[\frac{(\varepsilon-1)\,kz}{s+s_0}\frac{\theta''}{\psi''} + \frac{\varepsilon-1}{z} - \varepsilon'\right].$$

A sufficient condition for $\frac{\mathrm{d}\sigma_e}{\mathrm{d}y} > 0$ is $\varepsilon' \leq \frac{\varepsilon-1}{z}$. The expression for τ_y in (4.13) shows that this condition is also sufficient for $\frac{\mathrm{d}\tau_y}{\mathrm{d}\theta} > 0$. **QED**