Central Bank Digital Currency: Stability and Information*

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Abstract

We study how the introduction of a central bank digital currency changes the flow of information about current financial conditions to the central bank. We present a model in which banks have private information about the quality of their assets. Because they anticipate being bailed out in the event of a crisis, a moral hazard problem arises and regulation is potentially useful. However, the optimal regulatory policy is state-contingent and the policy maker observes the relevant state only with a lag. We show how, under some conditions, a central bank can infer the state more quickly by introducing a digital currency and monitoring the flow of funds into this new asset. This additional information increases the effectiveness of the central bank's regulatory policy and can thereby improve financial stability.

1 Introduction

There is growing interest in the possibility of central banks issuing digital currency, that is, a form of outside money that can be held electronically (as opposed to paper currency) and that is made widely available to firms and households. There has been substantial discussion in policy circles of the potential benefits of issuing a central bank digital currency (CBDC); see, for example BIS (2018) and the references therein. A variety of benefits have been discussed, including promoting more efficient payments and potentially improving the pass-through of monetary policy decisions.

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Many policy makers have nevertheless advocated a cautious approach, as both the benefits and potential costs of issuing a digital currency are still very uncertain.

We focus on what we believe is a novel aspect of this debate: issuing a CBDC may give the central bank more information about the state of the financial system and, in particular, about depositors' confidence in their banks. This additional information, in turn, may help the central bank more effectively regulate the banking system and, in doing so, improve financial stability. During periods of financial stress, commercial banks and other financial intermediaries have private information about both the quality of their assets and their liquidity position, that is, the willingness of their short-term creditors to continue to provide funding. A bank that is in weak position will often have an incentive to hide this fact from regulators, at least for a while, to avoid triggering some supervisory action(s). This combination of banks' private information and the incentive structure thus causes a delay in policy makers' response to an incipient financial crisis. Such a delay can increase both the likelihood of a full-blown crisis and the severity of such an event. A central bank digital currency could potentially affect the flow of information by providing banks' creditors with an additional liquid investment opportunity. In practice, this investment could be designed in a variety of different ways. For example, it could be either a deposit account at the central bank or a cryptographic token that is exchanged in a decentralized manner. We do not focus on the technological aspects of the currency here; for our purposes, it is sufficient to think of the central bank as directly offering deposit accounts to firms and households. Instead, our focus is on how the availability of this option affects the withdrawal behavior of banks' creditors in times of financial stress and how, in turn, this withdrawal behavior changes the information available to the central bank.

Suppose, for example, that a bank's short-term creditors learn that the quality of its assets has declined. Currently, if they wish to withdraw funding from the bank, they can shift their funds into another bank or into other liquid assets (for example, government bonds). These withdrawals from the bank might not be immediately observed by regulators and, even if they are, might be difficult to distinguish from the regular inflows and outflows generated by a bank's client transactions. Once a CBDC is introduced, in contrast, the central bank has a new source of information: the flow of funds into this digital currency. The design features of this currency, such as the interest rate it pays (if any), will determine how attractive it is to investors both in normal times and in periods of financial stress. Taking the relative attractiveness of the CBDC in different states into account, the central bank will be able to use information on inflows into the CBDC to more quickly infer the state of the financial system.

We base our analysis on a modified version of the classic model of Diamond and Dybvig (1983). As in the existing literature, private agents have an incentive to pool their resources in a bank to insure against idiosyncratic liquidity risk. In our setting, however, agents face the possibility of two distinct types of liquidity shocks. Some agents will be “impatient” and face an immediate consumption need, as is standard. Other agents will have access to an outside investment opportunity. These agents will desire the ability to withdraw from the bank, but at the same time will have the ability to invest in a CBDC if it is more attractive than their outside opportunity. The remaining agents are patient and need not withdraw from the bank, but will also have the option to withdraw and invest in the CBDC.

The value of a bank’s assets is stochastic and depends on the realization of one of two aggregate states. Banks and their investors privately observe this state at the beginning of the interim period. There is also a policy maker who has the ability to tax agents’ endowments and to provide a public good. The policy maker observes the state of nature only with a lag and can “bail out” banks by directly transferring some of this tax revenue to them in the bad aggregate state. The policy maker is unable to commit to the details of a bailout policy. As in Keister (2016), this inability to commit implies that the policy maker will give larger bailouts to banks that are in worse financial condition. Anticipating this reaction, banks have an incentive to pay out more funds to withdrawing creditors. The policy maker could limit these payments through regulation, but is unable to determine whether such limits are warranted until it can learn the state. As in Keister and Mitkov (2020), banks and the investors have an incentive to delay revealing their private information to the policy maker, in order to continue operating with fewer restrictions.

We introduce a CBDC into this environment and study how the design features of the currency, particularly the interest rate it pays, affect the incentive for bank creditors to shift funds into it, both in normal times and in periods of market stress. We begin by showing how a CBDC affects the efficient distribution of resources in the economy. We then derive conditions on the environment under which the policy maker is able to design the CBDC in such a way that it can learn the state of the financial system more quickly.

Some observers have discussed the possibility that a CBDC will increase financial fragility because it gives banks’ creditors a better option that they can “run into” at the first sign of trouble. According to this view, withdrawing funds from the banking system is currently costly. By providing a low-cost alternative to bank deposits, a CBDC would make bank creditors more likely to withdraw their funding and shift into the CBDC. As a result, this argument indicates that introducing a CBDC may increase the incidence of bank runs and financial crisis. These effects are present in our model, but we show that there is another side to the story. The incentives of bank creditors to withdraw funding also depends critically on what sort of policy response they expect if a run develops. If the CBDC also allows the policy maker to react more quickly to an incipient crisis, its net effect can be to increase rather than decrease financial stability.

The remainder of the paper is organized as follows. In Section 2, we describe the environment with no CBDC and in Section 3 we derive the efficient allocation of resources in this baseline environment. In Section 4, we derive the equilibrium in this baseline model and illustrate why the policy maker is slow to infer the state of the financial system in this equilibrium. In Section 5, we introduce a CBDC into the model and show how the central bank can use this new tool to more quickly infer the state. In Section 6, we offer some concluding remarks.

2 The environment

The economy consists of four periods, $t = 0, 1, 2, 3$. There is a single private consumption good in each period, as well as a public good which is consumed only in the last period.

There is a continuum of **banks** in each of two locations. These banks take deposits of the consumption good in period zero and invest in a long-term project that matures in period 2. The return on this project is $R(s)$, where $s \in \{h, \ell\}$ is an aggregate state with $R(h) = R > 1$ and $R(\ell) < R(h)$. It is possible that $R(\ell) < 1$ holds, so that the long term project is unproductive in the low state. The probability of the good state $s = h$ is denoted $1 - q$ while the probability of the bad state $s = \ell$ is q. Investment can be liquidated at no cost in period 1.

Investors are indexed by $i \in [0, 1]$ in each of the two different locations. Each investor is endowed with one unit of the good in period 0 and has preferences characterized by

$$U(c_1^i, c_2^i, c_3^i, g, \omega^i) \equiv u\left(\omega_1^i c_1^i + \omega_2^i c_2^i + (1 - \omega_1^i - \omega_2^i) c_3^i\right) + v(g)$$

where c_t^i is the private consumption of investor i in period t, and g is the consumption of the public good in period 2. The random variables $\omega_1^i, \omega_2^i \in \{0, 1\}$ are such that $\omega_1^i \omega_2^i = 0$. In other words, each investor only receives utility from consuming in one of the three periods. The probability that an investor wants to consume in the first period ($\omega_1^i = 1$) is $\pi\alpha$, the probability that the investor wants to consume in the second period ($\omega_2^i = 1$) is $1 - \pi$, and the probability that the investor wants to consume in the third period ($\omega_1^i + \omega_2^i = 0$) is $\pi(1 - \alpha)$.

Investors who consume in period 3 are special in that they have access to an outside investment opportunity that becomes available to them in period 1. Formally, a type 3 investor can invest in period 1 in a technology that returns $R_3 \in [\rho, \bar{\rho}]$ in period 3. This technology is the only way for those investors to transfer resources to period 3. In the absence of another option, they will use it even if its return is less than 1. For simplicity, we assume each type 3 investor draws the return R_3 from a uniform distribution on $[\rho, \bar{\rho}]$. For now neither ρ nor $\bar{\rho}$ depend on the aggregate state, but we will relax this assumption when we introduce CBDC. A law of large numbers implies that a measure π of investors will want to withdraw from their banks in period 1, of which a measure $\pi(1-\alpha)$ of investors will invest and consume in period 3. Also, the bank cannot distinguish between investors 1 and 3 and, therefore, pays the same amount $c_1(s)$ to both types when they withdraw.[1]

Finally, the economy has a **government** investing in the public good. The objective of the government is to maximize the sum of investors' expected utilities at all times. The government can only raise funds in period 0 by imposing a tax τ on investor's endowment. From these tax receipts, the government finances the public good and bails out banks if it deems it desirable. The state s is an aggregate state across all islands and it is private information to the private sector. So the government cannot observe the state of the economy,[2] the fraction of withdrawals, or the withdrawal amounts at any bank. However, the government has the right to observe a bank's balance sheet when this bank requests a bail-out or when it suspects that this bank is in trouble. The other parameters are common knowledge. The government cannot commit to the terms of a bail-out ex-ante, however and for simplicity we will assume the government can commit to providing no bailout if the state is h.

The timeline of the economy is as follows:

In period $t = 0$:

1. The government chooses the tax rate $\tau \geq 0$.
2. Investors deposit their after-tax endowment with the bank at their location
3. The government announces its regulatory policy, consisting of a bail-out amount as well as payments to deposit holders.
4. Banks choose their deposit contracts.

In period $t = 1$:

1. In each location, investors $i \in [0,1]$ observe their liquidity shock ω^i.
2. Investors and banks observe the aggregate state s.
3. Each investor chooses whether to withdraw and, if so, whether to invest in the $t = 3$ technology or CBDC when available.

[1] Type 1 investors do not play a critical role in our model, but we include them to preserve features of the well-known Diamond-Dybvig framework. Type 3 investors may choose to hold CBDC when the state is h (in normal times) if it is available, which creates a (simple) signal extraction problem for the central bank.

[2] Note that there is no trade in markets in this model, so no market prices are available to make inferences about the state.

4. Banks make payments to investors that choose to withdraw and/or decide to ask for a bail-out.

5. When a bank requests a bail-out, the government observes s and may choose to bail out the bank using the tax revenue. All unspent tax revenue is used to provide public good.

6. The government dictates the payment of a bank benefiting from a bail out (at $t = 2$)

In period $t = 2$ **and** $t = 3$, investors withdraw and/or consume the return from their investment. We make the following assumptions.

Assumption 1. *Preferences are represented by an increasing and concave utility function* $u(c)$ *such that* $cu'(c)$ *is decreasing, that is* $-u''(c).c/u'(c) > 1$*, and* $v(g) \geq 0$ *is increasing and strictly concave.*

Also, we assume that the investment technology of type 3 investors is good enough that their expected marginal utility of a payout in period 1 is always lower than the one for type 1 investors,

Assumption 2. ρ *and* $\bar{\rho}$ *are such that* $\int_{\rho}^{\bar{\rho}} R_3 u'(R_3 c) dR_3 \leq u'(c)$*.*

Since $cu'(c)$ is decreasing, a sufficient condition for Assumption 2 to hold is $\rho \geq 1$. In later sections, we will want to consider the consequences of having $\rho < 1$. We proceed as follows: First we solve for the constrained efficient allocations when the regulator cannot observe ω^i and is forced to give the same amount of resources to type 1 and 3 when they withdraw at date 1. We use c_{is} to denote the consumption of type i in state s, and b_s for the amount paid out of the tax receipt to type 2 investors. An allocation is a vector $(\tau, c_{is}, b_s)_{i \in [0,1], s \in \{h,\ell\}}$. We also assume that if one bank requests a bail-out then the regulator decides on the bailout amounts by considering the entire banking system.

3 Constrained Efficiency

Suppose the planner cannot observe the return R_3 specific to each type 3 investor. Then the payment to investors of type 1 and 3 must be the same and we denote it by c_{1s} with the understanding that investors of type 3 will invest this amount in their technology with a return of R_3. Then the constrained efficient allocation is a vector of consumption for each type of investor in each state c_{is}, a tax τ, and the bail-out in the low state b_ℓ that maximize the expected utility of an investor at $t = 0$ who faces a probability $\pi\alpha$ to become a type 1, $\pi(1 - \alpha)$ to become a type 3, and $1 - \pi$ to become a type 2, subject to the feasibility constraints, or

$$\max \quad \sum_{s=h,\ell} q_s \left[\pi \left[\alpha u(c_{1s}) + (1-\alpha) \int u(R_3 c_{1s}) dR_3 \right] + (1-\pi) u(c_{2s}) + v(\tau - b_s) \right]$$

subject to

$$\begin{aligned} (1-\pi) c_{2s} &= R_s \left[1 - \tau + b_s - \pi c_{1s} \right] \\ b_\ell &\geq 0 \quad \text{and } b_h = 0 \end{aligned}$$

Using the resource constraint to replace c_{2s}, the first order conditions with respect to c_{1s}, b_ℓ, and τ, are respectively

$$\begin{aligned} \left[\alpha u'(c_{1s}) + (1-\alpha) \int R_3 u'(R_3 c_{1s}) dR_3 \right] - R_s u'(c_{2s}) &= 0 \\ -v'(\tau - b_\ell) + R_\ell u'(c_{2\ell}) + \lambda_\ell &= 0 \\ (1-q) \left[v'(\tau) - R_h u'(c_{2h}) \right] + q \left[v'(\tau - b_\ell) - R_\ell u'(c_{2\ell}) \right] &= 0 \end{aligned}$$

where λ_ℓ is the Lagrange multiplier on $b_\ell \geq 0$. We can show some important properties of the constrained efficient allocation. All proofs are in the Appendix.

Proposition 1. *The constrained efficient allocation* $(c_{1s}^*, c_{2s}^*, b_\ell^*)$ *satisfies* $c_{1s}^* < c_{2s}^*$ *as well as* $b_\ell^* > 0$ *and*

$$(c_{1h}^*, c_{2h}^*) > (c_{1\ell}^*, c_{2\ell}^*) .$$

As in Diamond and Dybvig (1983), the constrained efficient allocation provides insurance to investors against the liquidity shock in both the high and low states. Since $R_h > R_\ell$, the allocation in the high state is greater than the allocation in the low states, for all types. However, if the planner invests all its tax receipt in the public good in the low state, investors of type 2 would consume too much of the public good relative to the consumption good. So the planner finds it optimal to also compensate investors for the low return in the low state by cutting its investment in the public good. In this sense, the efficient allocation also provides insurance across the two states.[3]

[3] Contrary to Diamond and Dybvig (1983), since $\tau > 0$ we may not have $c_{1h} > 1$ despite Assumption 1.

4 Implementation

We now study the implementation of the constrained efficient allocation by a continuum of banks. Each bank serves a representative measure of investors in separate locations. A bank accepts deposits in period 0 and invests in a set of identical projects. Each project requires one unit of investment in period 0 and returns R_s in period 2 in state s. The state is aggregate so all banks in all locations face the same return R_s.[4] The realization of the state is observed by banks and investors, but no one else. In particular, the regulator receives no information about the state.

4.1 The banking contract

After investors learn about their type and state s, they each inform their bank whether they want to withdraw in period 1 or 2. Investors who chose to withdraw in period 1 then begin arriving at the bank. No trade can occur among investors during this process. Investors can consume or invest the payment they receive from their bank, and they return in isolation. Notice that we do not assume sequential service and we allow the bank to suspend the convertibility of its deposits. Therefore, the bank first collects withdrawal demand in period 1 and then decides how much of it to satisfy.

A banking contract is a payment to investors who withdraw early in period 1, and the payment in period 2 is given by the bank's resource constraint, including bail-out payments from the regulator. Since banks cannot distinguish between type 1 and 3 investors who withdraw in period 1, banks will make the same payment c_{1s} as a function of the state and the number of withdrawal requests $w_{k,s} \in [\pi, 1]$.

The payment is restricted by the regulator to be in the set $\mathcal{X}$, independent of s since that information is not available to the regulator. In the baseline case, we require

$$\{0, c_{1h}^*, c_{1\ell}^*\} \in \mathcal{X}$$

First consider the equilibrium when $s = h$. When the state is $s = h$, the regulator commits to provide no bail-out. The strategy for type 1 and 3 investors is always to withdraw in period 1. Therefore, there are at least π withdrawals in period 1. We need to know if there can be more.

Let $y_{k,s}^i \in \{0, 1\}$ denote the strategy of investor i in bank k in state s, conditional on being

[4] We will concentrate on symmetric equilibria. If R_s could differ across islands, we could envisage another equilibrium where depositors would withdraw to deposit in banks located on another island. Since the long term investment can only be initiated in period 0, consistency requires that the bank accepting these deposits would promise at most the return on storage. Therefore, a regulator offering a slightly higher return than storage would attract those deposits thus obtaining the necessary information for a possible intervention.

a type 2. $y_s^i = 0$ means that the investor of type 2 wants to withdraw and $y_s^i = 1$ means that the investor does not want to withdraw. Whether the investor is able to withdraw depends on the payment strategy of their banks. The total demand for withdrawals in state s from type 2 investors in bank k is

$$x_{k,s} = \int \left(1 - y_{k,s}^i\right) di$$

and total withdrawal ***demand*** in state s in bank k is

$$w_{k,s} = \pi + (1-\pi) \int \left(1 - y_{k,s}^i\right) di$$

As mentioned above, the effective withdrawals (denoted $\bar{w}_{k,s}$) may be lower than the demand for withdrawals if the bank decides to suspend convertibility of its deposits at some point. For example although $w_{k,\ell} > \pi$ the bank may choose to only pay a fraction $\bar{w}_{k,\ell} < w_{k,\ell}$ of investors (selected at random).

An equilibrium is a profile y^* such that (i) given the strategies of investors at the other banks, $y_{k,s}^*$ is an equilibrium of the withdrawal game in state s in bank k, (ii) the contract in bank k maximizes the utility of investors in bank k, taking as given the contract and strategies of investors at the other banks, and (iii) the government bailout and resolution policy maximizes total welfare taking into account the contract and strategies y^*.

Consider first the contract when the state is $s = h$. In state h, there is an equilibrium contract that implements the constrained efficient allocation (c_{1h}^*, c_{2h}^*). The equilibrium contract in period 1 is simply

$$c_{1,h}(w_{k,h}) = \begin{cases} c_{1h}^* & \text{if } w_{k,h} \leq \pi \\ 0 & \text{otherwise} \end{cases}$$

while the equilibrium contract in period 2 is given by the resource constraint,

$$(1-\pi)c_{2,h}(w_{k,h}) = R_h \left[1 - \tau - \pi c_{1,h}(w_{k,h})\right]$$

This is the usual contract with suspension that implements the efficient allocation as a unique equilibrium with no run. Since $c_{2,h}^* > c_{1,h}^* \geq c_{1,h}(w_{k,h})$, it is a strictly dominating strategy for type 2 investors to not run. This contract maximizes the expected utility of investors in period 0, since it is the solution to the constrained planner's problem. It is key for the argument that the regulator can commit to not bail-out the bank in the high state.

Now consider the contract in the low state $s = \ell$. In this case, the regulator is ready to bail out banks but she only knows the state once a bank asks for a bail-out. Given the demand for withdrawal is $w_{k,\ell}$ and the bank chooses the effective number of withdrawals $\bar{w}_{k,\ell} \leq w_{k,\ell}$, the per-capita resources left in bank k after it paid $c_{1k}(w_{k,\ell})$ to a fraction $\bar{w}_{k,\ell}$ of depositors and after receiving a bailout of b_k is

$$\psi_k = \frac{1 - \tau + b_k - \bar{w}_{k,\ell} c_{1,k}(w_{k,\ell})}{1 - \bar{w}_{k,\ell}}.$$

While all type 1 and 3 investors want to withdraw, some of them may be left empty handed in period 1 when the bank suspends payments, that is $\bar{w}_{k,\ell} < w_{k,\ell}$. Let n_k be the fraction of type 1 and 3 investors who were not allowed to withdraw. By the law of large numbers, αn_k of this group will be of type 1 and $(1-\alpha)n_k$ will be of type 3. We derive the expression for n_k below, and given n_k the optimal allocation of funds after the bank receives a bail-out solves

$$V(\psi_k, n_k) = \max_{c_{1,b}, c_{2,b}} n_k \left[\alpha u(c_{1,b}) + (1-\alpha) \int u(R_3 c_{1,b}) dR_3 \right] + (1 - n_k) u(c_{2,b})$$

subject to

$$(1 - n_k) \frac{c_{2,b}}{R_\ell} + n_k c_{1,b} = \psi_k.$$

The solution to this problem, denoted $(c_{1,b}, c_{2,b})$, satisfies

$$\alpha u'(c_{1,b}) + (1-\alpha) \int R_3 u'(R_3 c_{1,b}) dR_3 = R_\ell u' \left(R_\ell \frac{\psi_k - n_k c_{1,b}}{1 - n_k} \right) \tag{1}$$

where $c_{2,b}$ is given by the resource constraint. Our assumptions imply $c_{2,b} > c_{1,b}$. The envelope condition gives the marginal value of resources in the bank,

$$\frac{\partial V(\psi_k, n_k)}{\partial \psi_k} = R_\ell u' \left(R_\ell \frac{\psi_k - n_k c_{1,b}}{1 - n_k} \right).$$

We can now solve for the optimal bailout amount in the low state. Assuming that all banks ask for a bailout the solution to the bailout amount $b_{\ell,k}$ solves

$$\max_{b_{\ell,k}} \quad (1 - \bar{\omega}_{k,\ell}) \int_k V(\psi_k, n_k) dk + v \left(\tau - \int_k b_{\ell,k} dk \right)$$

with first order condition (using the envelope condition above)

$$R_\ell u' \left(R_\ell \frac{\psi_k - n_k c_{1,b}^k}{1 - n_k} \right) = v' \left(\tau - \int_k b_{\ell,k} dk \right) \tag{2}$$

In state ℓ, the marginal utility of type 2 investors must be equalized across banks, so that

$$c^1_{2,b} = c^2_{2,b} = \hat{c}_{2,b}$$

Then (1) which describes the solution to the optimal allocation following a bail-out $b_{\ell,1}$ and b_{ℓ_2} implies that

$$c^1_{1,b} = c^2_{1,b} = \hat{c}_{1,b}$$

Therefore, following the bailout of one bank, the allocation given to the remaining patient investors in the other banks asking for a bail-out is unchanged. In particular, and importantly, the post-bailout allocation in bank k does not depend on the distribution of the resources of a particular bank k prior to the bail-out. The resource constraint post bailout is,

$$(1-n_k)\frac{\hat{c}_{2,b}}{R_\ell} + n_k\hat{c}_{1,b} = \psi_k = \frac{1-\tau+b_k-\bar{w}_{k,\ell}c_{1,k}(w_{k,\ell})}{1-\bar{w}_{k,\ell}} \tag{3}$$

Finally the fraction of remaining investors who are of type 1 or type 3 $n_k(w_\ell, \bar{w}_\ell)$, given w_ℓ and $\bar{w}_\ell$, is given by the strategy of type 2 investors in bank k

$$w_{k,\ell} = \begin{cases} \pi \text{ (no run)} \\ \in [\pi, 1] \\ 1 \text{ (run)} \end{cases} \Rightarrow \begin{cases} \bar{w}_{k,\ell} \leq \pi & \Rightarrow \quad n(\pi, \bar{w}_{k,\ell}) = \frac{\pi - \bar{w}_{k,\ell}}{1-\bar{w}_{k,\ell}} \\ \bar{w}_{k,\ell} \leq w_{k,\ell} & \Rightarrow \quad n(w_{k,\ell}, \bar{w}_{k,\ell}) = \frac{\pi}{w_{k,\ell}}\frac{w_{k,\ell}-\bar{w}_{k,\ell}}{1-\bar{w}_{k,\ell}} \\ \bar{w}_{k,\ell} \leq 1 & \Rightarrow \quad n(1, \bar{w}_{k,\ell}) = \pi \end{cases}$$

Notice that n_k is strictly decreasing in $\bar{w}_{k,\ell}$, so that the more investors are served before the bail-out, the less type 1 and type 3 investors who remain to be served after the bail-out in bank k. Also the bail-out amount increases with the payment to withdrawers $c_{1,k}(w_{k,\ell})$ and the fraction of investors served. These comparative statics give an indication that a bank trying to maximize the bail-out amount it receives from the regulator will pay a large amount to many depositors in period 1.[5]

[5] If there is no run $w_{k,\ell} = \pi$ and $n(\pi, \bar{w}_{k,\ell}) = \frac{\pi - \bar{w}_{k,\ell}}{1-\bar{w}_{k,\ell}}$ so (3) becomes

$$(1-\pi)\frac{\hat{c}_{2,b}}{R_\ell} + (\pi - \bar{w}_{k,\ell})\hat{c}_{1,b} = \psi_k = 1-\tau+b_k-\bar{w}_{k,\ell}c_{1,k}(\pi)$$

So the bail-out amount is increasing in the payment to depositors pre-bail out $c_{1,k}(\pi)$, and in the fraction of investors served $\bar{w}_{k,\ell}$ whenever the bank paid its depositors more before the bail-out than the amount it is required to pay post-bail-out, $c_{1,k}(\pi) > \hat{c}_{1,b}$. If there is a full run $w_{k,\ell} = 1$ and $n(\pi, \bar{w}_{k,\ell}) = \pi$ so (3) becomes

$$(1-\pi)\frac{\hat{c}_{2,b}}{R_\ell} + \pi\hat{c}_{1,b} = \psi_k = \frac{1-\tau+b_k-\bar{w}_{k,\ell}c_{1,k}(1)}{1-\bar{w}_{k,\ell}}$$

4.2 Optimal strategy of type 2 investors

After a fraction $w_{k,\ell}$ of investors place a withdrawal order with bank k, this bank will choose to serve $c_{1,k}(w_{k,\ell})$ to a fraction $\bar{w}_{k,\ell} \leq w_{k,\ell}$. The bank then asks for a bail-out and is placed in resolution (it is possible at this stage that $b = 0$). Given this policy, type 2 investors will adopt the following strategy: They will place a withdrawal order whenever what they receive in period 1 $c_{1,k}(w_{k,\ell})$ is greater than $\hat{c}_{2,b}$ and they won't otherwise. Hence,

$$w_{k,\ell} = \begin{cases} \pi & \text{if } c_{1,k} < \hat{c}_{2,b} \\ \in [\pi, 1] & \text{if } c_{1,k} = \hat{c}_{2,b} \\ 1 & \text{if } c_{1,k} > \hat{c}_{2,b} \end{cases} \Longrightarrow \begin{cases} n(\pi, \bar{w}_\ell) = \frac{\pi - \bar{w}_\ell}{1 - \bar{w}_\ell} \\ n(w_\ell, \bar{w}_\ell) = \frac{\pi}{w_\ell} \frac{w_\ell - \bar{w}_\ell}{1 - \bar{w}_\ell} \\ n(1, \bar{w}_\ell) = \pi \end{cases} \tag{4}$$

We have the following result.

Proposition 2. *If $(w_{k,\ell}, c_{1,k})$ satisfy (4) then there is a banking contract $c_{1,k}(w_{k,\ell})$ that implements this allocation as the unique equilibrium of the withdrawal game played by investors in the low state.*

In the sequel, we show that, optimally, the bank will set $\bar{w}_{k,\ell}$ as follows: If $w_{k,\ell} \leq \pi$ then $\bar{w}_{k,\ell} = w_{k,\ell}$ and if $w_{k,\ell} > \pi$ then $\bar{w}_{k,\ell} = \min\{w_{k,\ell}, (1-\tau)/c_{1,k}\}$. So the bank always tries to serve as many investors as possible, until it runs out of resources and then it asks for a bail-out. Bank k's optimal contract problem is to choose $(w_{k,\ell}, \bar{w}_{k,\ell}, c_{1,k})$ to maximize

$$\begin{aligned} V_k(w_{k,\ell}, \bar{w}_{k,\ell}, c_{1,k}) \quad = \quad & \mathbb{I}_{w_{k,\ell} > \pi} \left\{ \begin{array}{c} \bar{w}_{k,\ell} \left[\left(\frac{\pi}{w_{k,\ell}} \alpha + 1 - \frac{\pi}{w_{k,\ell}} \right) u(c_{1,k}) + \frac{\pi}{w_{k,\ell}} (1-\alpha) \int u(R_3 c_{1,k}) dR_3 \right] \\ + (1 - \bar{w}_{k,\ell}) \left[n_k \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha) \int u(R_3 \hat{c}_{1,b}) dR_3 \right] + (1 - n_k) u(\hat{c}_{2,b})) \right] \end{array} \right\} \\ & + \\ & \mathbb{I}_{w_{k,\ell} \leq \pi} \left\{ \begin{array}{c} \bar{w}_{k,\ell} \left[\alpha u(c_{1,k}) + (1-\alpha) \int u(R_3 c_{1,k}) dR_3 \right] \\ + (1 - \bar{w}_{k,\ell}) \left[n_k \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha) \int u(R_3 \hat{c}_{1,b}) dR_3 \right] + (1 - n_k) u(\hat{c}_{2,b})) \right] \end{array} \right\} \end{aligned}$$

where $c_{1,k} \in \mathcal{X}$, subject to the implementability constraint and $\bar{w}_{k,\ell} \leq w_{k,\ell}$. We first show that the bank's optimal disbursement policy is to satisfy all the demand of withdrawals, or to run out of resources, whichever comes first.

Lemma 1. *Given $w_{k,\ell}$ and $c_{1,k}$, the optimal disbursement policy when expecting a bailout is $\bar{w}_{k,\ell} = \min\{w_{k,\ell}, (1-\tau)/c_{1,k}\}$.*

The bank chooses $\bar{w}_{k,\ell}$ to trade off the benefit of serving more investors early on relative to serving them in a bail-out, and the effect on the bail-out value the bank receives of serving more

so again the higher the payment made to depositors $c_{1,k}(1)$ the higher the bail-out amount, and the higher the fraction of investors served $\bar{w}_{k,\ell}$ the bigger the bail-out if the payout has been large, $c_{1,k}(1) \geq 1 - \tau + b_k$.

investors earlier on. Serving more investors earlier reduces the resources of the banks, but since the bail-out allocation is independent of the bank's resources, the bank does not internalize this effect. The bank however recognizes that serving more investors earlier reduces the number of type 1 and 3 investors remaining to be served after the bail-out. Since those investors receive less than type 2 investors in a bail-out, the bank finds optimal to serve as many investors as possible before it calls for a bail-out.

Next suppose $w_{k,\ell} > \pi$, so that there is a run, if only a partial one. The next Lemma states the optimal disbursement policy in this case,

Lemma 2. *Given $w_{k,\ell} > \pi$, there is $\tilde{c}_1$ such that the optimal disbursement policy is $\mathring{c}_1$ where $\mathring{c}_1 = \max \mathcal{X}$ if $\max \mathcal{X} \leq \tilde{c}_1$, and $\mathring{c}_1 = \tilde{c}_1$ otherwise. In any case, $\bar{w}_\ell = \min \{w_{k,\ell}, (1-\tau)/\mathring{c}_1\}$.*

Lemma 2 shows that the bank sets $c_{1,k}$ to maximize the disbursement when there is a run. The consumption is set at the highest possible amount allowed by the regulator, or the amount that extinguishes all of the bank's resources. We would expect that the bank accommodates all the withdrawal demand $\bar{w}_\ell = w_{k,\ell}$ whenever $\mathring{c}_1 = \max \mathcal{X}$: The bank would like to set a higher consumption level but doing so would signal that the bank is trying to game the system. So instead of increasing further the amount paid out, the bank increases the number of depositors it serves.

We now characterize the bank's optimal payout in the equilibrium where there is no run. We will assume for simplicity that $\pi < (1-\tau)/\hat{c}_{2,b}$ (so paying $\hat{c}_{2,b}$ to a measure π of investors does not extinguish resources.)

Lemma 3. *Let $\pi \hat{c}_{2,b} \leq 1-\tau$. Given $w_{k,\ell} = \pi$, the optimal disbursement policy is $c_1 = \hat{c}_{2,b}$ if $\hat{c}_{2,b} \in \mathcal{X}$, and c_1 is the highest element in $\mathcal{X}$ that is less than $\hat{c}_{2,b}$ otherwise. In any case, $\bar{w}_{k,\ell} = \pi$.*

Lemma 3 characterizes the bank's payout policy conditional on being in a no-run equilibrium. Given the equilibrium does not have a run, it must be that the bank pays at most $\hat{c}_{2,b}$ to the fraction π of investors withdrawing in period 1. The lemma shows that the bank will choose to pay the maximum it can to those investors, so $\hat{c}_{2,b}$.

We now combine Lemma 2 and Lemma 3 to obtain the solution for the optimal allocation of a bank. The bank will choose the allocation inducing a run whenever

$$V_k(1, \mathring{c}_{1,k}) > V_k(\pi, \hat{c}_{2,b})$$

and it will choose the allocation inducing no run otherwise. Notice that the bank benefits from a bailout in both cases. From the definition of $V_k(1, \mathring{c}_{1,k})$ and $V_k(\pi, \hat{c}_{2,b})$ we obtain that bank k

selects the allocation that induces a run whenever[6]

$$H(\mathring{c}_{1,k}, \hat{c}_{1,b}, \hat{c}_{2,b}) \equiv \frac{1-\tau}{\mathring{c}_{1,k}} \left[\pi \left(h(\mathring{c}_{1,k}) - h(\hat{c}_{1,b})\right) + (1-\pi)\left[u(\mathring{c}_{1,k}) - u(\hat{c}_{2,b})\right]\right] - \pi \left\{h(\hat{c}_{2,b}) - h(\hat{c}_{1,b})\right\} > 0$$

where we used $h(c) = \alpha u(c) + (1-\alpha)\int u(R_3 c) dR_3$. We thus have the following result.

Proposition 3. *Assume $\pi \hat{c}_{2,b} < 1-\tau$. A bank k in the low state adopts the following strategy,*

(i) If $H(\mathring{c}_{1,k}, \hat{c}_{1,b}, \hat{c}_{2,b}) > 0$, the equilibrium allocation is $w_{k,\ell} = 1$, and $c_{1,k} = \mathring{c}_{1,k}$. The bank serves $(1-\tau)/\mathring{c}_{1,k}$ investors before asking for a bail-out.

(ii) If $H(\mathring{c}_{1,k}, \hat{c}_{1,b}, \hat{c}_{2,b}) < 0$, the equilibrium allocation is $w_{k,\ell} = \pi$, and $c_{1,k} = \hat{c}_{2,b}$. The bank serves π investors before asking for a bail-out.

(iii) If $H(\mathring{c}_{1,k}, \hat{c}_{1,b}, \hat{c}_{2,b}) = 0$, there are two equilibria, one with $w_{k,\ell} = 1$, and $c_{1,k} = \mathring{c}_{1,k}$ and another one with $w_{k,\ell} = \pi$, and $c_{1,k} = \hat{c}_{2,b}$.

Since the regulator does not learn about the state before a bank calls for a bailout, the bank will serve as many withdrawing depositors as possible, while maxing out on its resources when there is a run.

4.3 Musical chairs

The proof of Proposition 3 shows that if all banks choose $w_{k,\ell}$ as in Proposition 3, then a bank has no incentive to choose $\bar{w}_{k,\ell} \leq w_{k,\ell}$. This means that banks do not have any incentive to inform the regulator about the state before any other bank does. So there is a sense in which "as long as the music is playing, banks get up and dance." Also, we show that if $c^*_{1,\ell} \in \mathcal{X}$ and all banks but one chooses $c^*_{1,\ell}$ and do not ask for a bailout then it is optimal for the remaining bank to choose an allocation higher than $c^*_{1,\ell}$. One such allocation is of course $c^*_{1,h} \in \mathcal{X}$. The bank does not have

[6] We have

$$\begin{aligned} V_k(1, \mathring{c}_{1,k}) &= \frac{1-\tau}{\mathring{c}_{1,k}} \left[\begin{array}{c} \pi\alpha\left[u(\mathring{c}_{1,k}) - u(\hat{c}_{1,b})\right] + (1-\pi)\left[u(\mathring{c}_{1,k}) - u(\hat{c}_{2,b})\right] \\ +\pi(1-\alpha)\int \left[u(R_3\mathring{c}_{1,k}) - u(R_3\hat{c}_{1,b})\right] dR_3 \end{array} \right] \\ &\quad + \pi\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b}) dR_3\right] + (1-\pi)u(\hat{c}_{2,b}) \end{aligned}$$

while assuming $\pi < (1-\tau)/\hat{c}_{2,b}$,

$$\begin{aligned} V_k(\pi, \hat{c}_{2,b}) &= \pi\left\{\left[\alpha u(\hat{c}_{2,b}) + (1-\alpha)\int u(R_3\hat{c}_{2,b}) dR_3\right] - \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b}) dR_3\right]\right\} \\ &\quad + \pi\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b}) dR_3\right] + (1-\pi)\, u(\hat{c}_{2,b})) \end{aligned}$$

to pick $\mathring{c}_{1,k}$ or $\hat{c}_{2,b}$ as derived above. Indeed, if a bank is the only bank asking for a bailout then the bailout allocation is $(c^*_{1,\ell}, c^*_{2,\ell})$; The reason is that the bank has measure zero, so the bail out policy is $\hat{c}_{1,b}$ and $\hat{c}_{2,b}$ such that

$$\begin{aligned} R_\ell u'(\hat{c}_{2,b}) &= v'(\tau) \\ \alpha u'(\hat{c}_{1,b}) + (1-\alpha)\int R_3 u'(R_3\hat{c}_{1,b})dR_3 &= v'(\tau) \end{aligned}$$

which gives $\hat{c}_{t,b} = c^*_{t,\ell}$ for $t = 1, 2$. Since bank k can "freely" extract resources from the public good, it is optimal for that bank to pay more than $c^*_{1,\ell}$ to investors asking for an early withdrawal. So paying $(c^*_{1,\ell}, c^*_{2,\ell})$ when all other banks are also paying $(c^*_{1,\ell}, c^*_{2,\ell})$ cannot be an equilibrium. The equilibrium is inefficient in the low state because banks can always choose $c^*_{1,h}$, which has to be an option for the equilibrium to be efficient in the high state. The inefficiency is reinforced here relative to Keister and Mitkov (2020) because banks can deplete all of their resources before asking for a bailout.

5 Central Bank Digital Currency

Suppose now the central bank (the regulator) offers investors access to its balance sheet in the form of an account based CBDC. Any investors can deposit resources at the central bank remunerated at the gross interest rate $1 + r$. We assume it is costly to pay interest rate on CBDC as the regulator has to finance it using the tax receipt τ. So remunerating CBDC comes at the cost of a lower provision of the public good.

In period 1, we envisage the following timing: after investors learn about their type and s, they each inform their bank whether they want to withdraw in period 1 or 2. Investors who chose to withdraw in period 1 then begin arriving at the bank. No trade can occur among investors during this process. Once investors place their withdrawal orders, banks choose the measure of investors to serve and they randomly select those investors from the pool of investors who placed a withdrawal order and serve them in sequence. Investors of type 1 will consume what they get from the bank. Investors of type 2 or 3 can invest it in CBDC or in their technology (for investors of type 3) and then they return in isolation. This means that the regulator will observe a sequence of deposits into CBDC originating from a particular bank and can decide to intervene based on this information. To be sure, the bank does not observe the type of investors using CBDC. Investors can consume the payment they receive from their bank, or from the regulator.

The relative measure of type 1 and type 3 investors can now differ across states. In state s the measure of type 3 investors is $(1-\alpha_s)\pi$. Also we assume those investors draw the return on their technology from a state dependent distribution, F_h with support $[\underline{R}_h, \bar{R}_h]$ in the high state and F_ℓ with support $[\underline{R}_\ell, \bar{R}_\ell]$ in the low state. Since CBDC deposits pay $1+r$, all type 3 investors with $R_3 < 1+r$ will use CBDC. Therefore, in the absence of runs, fewer investors of type 3 will use CBDC in the low state than in the high state whenever $(1-\alpha_\ell)F_\ell(1+r) \leq (1-\alpha_h)F_h(1+r)$. If this last inequality holds, a regulator who can only observe deposits into CBDC will not necessarily become suspicious when more than $(1-\alpha_\ell)F_\ell(1+r)$ investors are using CBDC, since this is the case in state h.

In order to maintain the results we derived earlier, we assume that the aggregate marginal utility of type 1 and 3 investors from any payment c in period 1 is higher in the high state than in the low state,

Assumption 3. *α_ℓ, α_h and F_h, F_ℓ are such that for all $c \geq 0$,*

$$z_h'(c) \equiv \alpha_h + (1-\alpha_h)\int_{\underline{R}_h}^{\bar{R}_h} R_3 u'(R_3 c) dF_h(R_3) \geq \alpha_\ell + (1-\alpha_\ell)\int_{\underline{R}_\ell}^{\bar{R}_\ell} R_3 u'(R_3 c) dF_\ell(R_3) \equiv z_\ell'(c).$$

This assumption implies that the weighted marginal utility of early withdrawers in the high state is higher than in the low state. So the optimal allocation will pay more to type 1 and 3 investors in the high state than in the low state. Our numerical example below shows that this assumption is sufficient but not necessary to obtain $c_{1h}^* > c_{1\ell}^*$.

5.1 The constrained efficient allocation with CBDC

The constrained efficient allocation with CBDC solves

$$\max \quad \sum_{s=h,\ell} q_s \left\{ \pi \left[\alpha_s u(c_{1s}) + (1-\alpha_s) \int u(\tilde{R}_3 c_{1s}) f_s(R_3) dR_3 \right] \right. \\ \left. + (1-\pi) u(c_{2s}) + v(\tau - b_s - \gamma_s(r) c_{1s}) \right\}$$

where $\tilde{R}_3 = \max\{1+r, R_3\}$, and $\gamma_s(r) = \pi(1-\alpha_s) r F_s(1+r)$ is the payment to the measure of investors 3 using CBDC per unit deposited, subject to

$$\begin{aligned} (1-\pi)c_{2s} &= R_s \left[1 - \tau + b_s - \pi c_{1s}\right] \\ b_\ell &\geq 0 \quad \text{and } b_h = 0 \end{aligned}$$

Notice that the bank pays $c_{1,s}$ to both type 1 and type 3 investors, and if type 3 investors use CBDC, the regulator pays the promised interest rate out of its tax receipts. In the Appendix we show that it is optimal to set $r^* = 0$ when the minimum return for the technology used by type 3 investors is large enough in both states. So $r^* = 0$ when $\underline{R}_h$ and $\underline{R}_\ell$ are sufficiently high. Since the weighed marginal utility of early withdrawers in the high state is higher than the one in the low state, we show the (constrained) efficient allocation pays more to early withdrawers in the high state than in the low state, $c^*_{1h} \geq c^*_{1\ell}$.

Lemma 4. *The constrained efficient allocation when CBDC pays $r^* = 0$ is characterized by $c^*_{1h} > c^*_{1\ell}$. If $\underline{R}_h = \underline{R}_\ell > 1$ then $r^* = 0$.*

Lemma 4 is useful for our purpose as it implies that in the bad state, banks mimicking the optimal disbursement in the good state will pay depositors too much relative to the optimum. In the sequel, we assume the regulator sets $\{0, c^*_{1h}(r), c^*_{1\ell}(r)\} \in \mathcal{X}$.

Notice first that if there is a run, so that $w_{k,\ell} > \pi$ and the disbursement c_1 is not c^*_{1h}, then the regulator will know that a run is under way as soon as an investor makes a deposit in CBDC. This is new relative to cash withdrawal and no CBDC. When investors withdraw cash, the regulator can only learn the aggregate withdrawals of cash from the bank, as it usually does not have access to granular withdrawal information. In other words, with cash only and given a large amount of withdrawals $\$X = n \times c_1$, the regulator learns $\$X$ but does not know if the number of withdrawers n is high or the amount withdrawn $\$c_1$ is high. With CBDC, a bank that wants to game the system has to pay the same amount in the good or the bad state, c^*_{1h} since the regulator would otherwise learn the bank's strategy in almost real time. In this sense, CBDC is useful to restrict the banks' disbursement policy in the low state. We assume from now on that the disbursement in the low state is either c^*_{1h} or $c^*_{1\ell}$. Since the bank only tries to game the system when it pays c^*_{1h} in the low state, this is the disbursement we concentrate on.

5.2 Preliminary considerations for remunerating CBDC

We analyze the circumstances under which the regulator can learn the state only by the number of interim deposits in CBDC. Investors of type 2 will only use CBDC if $r \geq 0$, as otherwise they will prefer storage.[7] Paying a negative rate on CBDC may slow down information acquisition by the regulator and she can only learn a run is underway if she remunerates well enough CBDC deposits.

However, remunerating CBDC deposits creates an inefficiency in state h. In that state, a measure $\pi(1-\alpha_h)F_h(1+r)$ of type 3 investors have a lower return on their technology than the

[7]In the presence of another asset such as a government bond, the return on CBDC has to be at least as high as the return of the bond net of the transaction cost involved in acquiring these bonds.

interest paid by CBDC and they will withdraw $c^*_{1,h}$ and use CBDC instead of their technology. So the payout to CBDC account holders from tax receipts is $r\pi(1-\alpha_h)c^*_{1,h}F_h(1+r)$. As a result the regulator will want to set r as low as possible in order to minimize this distortion. Also, type 2 investors will withdraw in the high state whenever $(1+r)c^*_{1,h} > c^*_{2,h}$, triggering a run in the high state. Hence, the interest rate on deposits cannot be too large,

$$1 + r \leq c^*_{2,h}/c^*_{1,h}.$$

In the next section, we derive banks' optimal choices given the regulator's policy.

5.3 Optimal policy

Now we turn to finding the optimal policy of banks in the low state, and the bailout allocation when banks paid $c^*_{1,h}$ instead of $c^*_{1,\ell}$ in the low state. We proceed as in the previous section.

Out of the measure $(1-\bar{\omega}_{k,\ell})$ of investors who were not served, we let n_k be the fraction of investors who are type 1 or 3 (see the expression for n_k below). By the law of large number, $\alpha_\ell n_k$ is of type 1 and $(1-\alpha_\ell)n_k$ is of type 3. Therefore, the optimal allocation of funds **post-bail out** solves

$$V(\psi_k, n_k) = \max_{c_{1,b},c_{2,b}} n_k \left[\alpha_\ell u(c_{1,b}) + (1-\alpha_\ell)\int u(\tilde{R}_3 c_{1,b})dR_3\right] + (1-n_k)u(c_{2,b})$$

subject to

$$(1-n_k)\frac{c_{2,b}}{R_\ell} + n_k c_{1,b} = \psi_k = \frac{1-\tau+b_k-\bar{w}_{k,\ell}c^*_{1,h}}{1-\bar{w}_{k,\ell}}$$

with the understanding that $\tilde{R}_3 = \max\{(1+\bar{r}), R_3\}$. The solution $(c_{1,b}, c_{2,b})$ satisfies

$$\alpha_\ell u'(c_{1,b}) + (1-\alpha_\ell)\int \tilde{R}_3 u'(\tilde{R}_3 c_{1,b})dR_3 = R_\ell u'\left(R_\ell\frac{\psi_k - n_k c_{1,b}}{1-n_k}\right) \tag{5}$$

where $c_{2,b}$ is given by the resource constraint. Our assumption implies $c_{2,b} > c_{1,b}$. The envelope condition gives

$$\frac{\partial V(\psi_k, n_k)}{\partial \psi_k} = R_\ell u'\left(R_\ell\frac{\psi_k - n_k c^k_{1,b}}{1-n_k}\right)$$

Assume that a bank asks for a bailout, then the solution to the bailout amount $b_{\ell,k}$ solves

$$\max_{b_{\ell,k}} \quad (1-\bar{\omega}_{k,\ell})\int_k V(\psi_k, n_k)dk + v\left(\tau - \int_k b_{\ell,k}dk - rc^*_{1,h}\int_k \eta(\bar{w}_{k,\ell}, r)dk\right)$$

where $\eta(\bar{w}_{k,\ell}, r)$ is the measure of investors from bank k who use CBDC after withdrawing in period 1, to be determined below. Notice that the regulator does not pay interest to investors using CBDC once a bailout is underway, but only to those investors who used CBDC before the bailout. The first order condition (using the envelope condition above)

$$R_\ell u'\left(c^k_{2,b}\right) = v'\left(\tau - \int_k b_{\ell,k} dk - r c^*_{1,h} \int_k \eta(\bar{w}_{k,\ell}, r) dk\right) \tag{6}$$

So in state ℓ, the marginal utility of investors 2 are again equalized across banks, so that

$$c^1_{2,b} = c^2_{2,b} = \hat{c}_{2,b}$$

Since more resources are paid out to CBDC, holding everything else constant (including $b_{\ell,k}$), concavity and the right-hand side of (6) implies $\hat{c}_{2b}$ is decreasing with r. Thus there is a threshold level for the interest rate r such that $(1+r)c^*_{1h} > \hat{c}_{2,b}$ for higher rates than this threshold. In this case, type 2 investors run in the low state. This lends credence to the idea that paying a higher interest rate on CBDC may generate instability. However, this argument is misleading if only slightly because the bailout amount $b_{\ell,k}$ is endogenous, and reacts to r. Still it is true that in our environment instability generates information on which the regulator can act.

The solution to the optimal allocation following a bail-out $b_{\ell,1}$ and $b_{\ell,2}$ then implies

$$c^1_{1,b} = c^2_{1,b} = \hat{c}_{1,b}$$

Therefore, following the bailout of one bank, the allocation for investors in other banks requesting a bailout is the same. In particular, the post-bailout allocation in bank k does not depend on the prior distribution of the resources of a particular bank k. Therefore b_k is given by

$$(1-n_k)\frac{\hat{c}_{2,b}}{R_\ell} + n_k \hat{c}_{1,b} = \psi_k = \frac{1-\tau+b_k-\bar{w}_{k,\ell}c^*_{1,h}}{1-\bar{w}_{k,\ell}}$$

Finally, as in the previous section, we determine the fraction $\bar{w}_{k,\ell}$ of investors who can be served before the bank calls to receive a bailout, the fraction of remaining investors who are of type 1 or type 3, $n_k(w_\ell, \bar{w}_\ell)$, and the fraction of investors who use CBDC $\eta(\bar{w}_{k,\ell}, r)$ after being served in period 1. Suppose a fraction $\xi \in [0,1]$ of investors of type 2 run. Then the measure of investors placing a withdrawal order is $w_\ell = \pi + \xi(1-\pi)$. When the bank decides to serve a measure $\bar{w} \leq w_\ell$

of investors, the measure of investors who are served is (by type)

$$\frac{\bar{w}}{w_\ell} \times \begin{cases} \xi(1-\pi) & \text{are type 2 (and use CBDC)} \\ \pi(1-\alpha_\ell)F_\ell(1+r) & \text{are type 3 and use CBDC} \\ \pi(1-\alpha_\ell)\left[1-F_\ell(1+r)\right] & \text{are type 3 and don't use CBDC} \\ \pi\alpha_\ell & \text{are type 1 (and don't use CBDC)} \end{cases}$$

Therefore, the measure of investors using CBDC prior to a bailout is

$$\eta(\bar{w}, r) = \frac{\bar{w}}{w_\ell}\left[\xi(1-\pi) + \pi(1-\alpha_\ell)F_\ell(1+r)\right].$$

The bank can only serve so many investors before it is caught by the regulator. Precisely, it must be that the bank's policy implies a lower amount of deposit with the CBDC than in the high state. So $\bar{w}$ must be such that

$$\frac{\bar{w}}{w_\ell}\left[\xi(1-\pi) + \pi(1-\alpha_\ell)F_\ell(1+r)\right] \leq \pi(1-\alpha_h)F_h(1+r).$$

On the left-hand side is the measure of investors served who will use CBDC when the bank serves $\bar{w}_{k,\ell}$ investors. On the right-hand side is the measure of investors who use CBDC in the high state. Hence the bank can serve *at most* σ_ℓ investors where

$$\sigma_\ell \equiv \frac{w_\ell \pi(1-\alpha_h)F_h(1+r)}{\xi(1-\pi) + \pi(1-\alpha_\ell)F_\ell(1+r)}$$

Using $w_\ell = \pi + \xi(1-\pi)$ to replace for ξ we obtain

$$\sigma_\ell(w_\ell, r) \equiv \frac{w_\ell \pi(1-\alpha_h)F_h(1+r)}{w_\ell - \pi + \pi(1-\alpha_\ell)F_\ell(1+r)}$$

To summarize, given w_ℓ the fractions $\bar{w}_\ell$ and $n_k(w_\ell, \bar{w}_\ell)$ are given by the strategy of type 2 agents in bank k

$$w_\ell = \begin{cases} \pi \text{ (no run)} & \text{if } (1+r)c^*_{1,h} < \hat{c}_{2,b} \\ \in [\pi, 1] & \text{if } (1+r)c^*_{1,h} = \hat{c}_{2,b} \\ 1 \text{ (run)} & \text{if } (1+r)c^*_{1,h} > \hat{c}_{2,b} \end{cases} \Rightarrow \begin{cases} \bar{w}_\ell = \pi & \Rightarrow \quad n(\pi, \bar{w}_\ell) = 0 \\ \bar{w}_\ell \leq \sigma_\ell(w_\ell, r) & \Rightarrow \quad n(w_\ell, \bar{w}_\ell) = \frac{\pi}{w_\ell}\frac{w_\ell - \bar{w}_\ell}{1-\bar{w}_\ell} \\ \bar{w}_\ell \leq \sigma_\ell(1, r) & \Rightarrow \quad n(1, \bar{w}_\ell) = \pi \end{cases}$$

A few aspects are worth pointing out.

1. If $c^*_{1,h} < \hat{c}_{2,b}$ then CBDC should only pay a marginal interest rate (if any) to induce as few type 3 investors as possible to use CBDC: This is enough to discipline the bank to only pay $c^*_{1,h}$ and a higher interest rate could trigger a run from investors of type 2.

2. If $F_h(\cdot)$ FOSD $F_\ell(\cdot)$, then $\sigma_{k,\ell}(1,r)$ is increasing in r and the bank can serve more investors before asking for a bailout.[8]

3. The maximum measure of investors served $\sigma_\ell(w_\ell, r)$ is less than π whenever

$$\frac{1-(1-\alpha_h)F_h(1+r)}{1-(1-\alpha_\ell)F_\ell(1+r)} \geq \frac{\pi}{w_\ell},$$

   that is the fraction of investors who are not using CBDC in the high state relative to the one in the low state should be high enough.[9] The intuition is straightforward: Without a run, only type 3 investors with a bad return on storage should use CBDC. Suppose this fraction is low. When investors run, the bank will serve some investors of type 2 and all those investors will then use CBDC. This may turn out to be a larger measure of investors than in normal times. So the cheating bank cannot serve many investors if there are a lot of type 2 investors before the regulator discovers it is indeed cheating. Notice that $\sigma_{k,\ell} \leq \pi$ does not mean there is no run. It means that CBDC limits the extent of the cheating, since the bank could ask for a bailout before it runs out of resources (when $\sigma_{k,\ell}(1,r)c^*_{1,h} < 1-\tau$ and notice that $\pi c^*_{1,h} < 1-\tau$).

Notice that n_k is strictly decreasing in $\bar{w}_{k,\ell}$, so the more investors are served before the bail-out, the less type 1 and type 3 remain to be served after the bail-out in bank k.

[8]

$$\frac{\partial \sigma_{k,\ell}(1,r)}{\partial r} = \frac{\pi(1-\alpha_h)f_h(1+r)\left\{(1-\pi)+\pi(1-\alpha_\ell)f_\ell(1+r)\left[\frac{F_\ell(1+r)}{f_\ell(1+r)} - \frac{F_h(1+r)}{f_h(1+r)}\right]\right\}}{1-\pi+\pi(1-\alpha_\ell)F_\ell(1+r)}.$$

[9]Since

$$\begin{aligned} \frac{w_\ell \pi(1-\alpha_h)F_h(1+r)}{w_\ell - \pi + \pi(1-\alpha_\ell)F_\ell(1+r)} &\leq \pi \\ w_\ell \frac{1-(1-\alpha_h)F_h(1+r)}{1-(1-\alpha_\ell)F_\ell(1+r)} &\geq \pi \end{aligned}$$

This is

$$w_\ell - \pi \geq w_\ell \frac{[(1-\alpha_h)F_h(1+r)-(1-\alpha_\ell)F_\ell(1+r)]}{1-(1-\alpha_\ell)F_\ell(1+r)}$$

Using the result of Lemma 2 and Lemma 3 in the previous section, we obtain that given $w_{k,\ell} \geq \pi$, the optimal disbursement policy is c_{1h}^* and $\bar{w}_\ell = \sigma_{k,\ell}(w_{k,\ell}, r)$ whenever $\sigma_{k,\ell}(1,r)c_{1,h}^* < 1 - \tau$. The main differences are that (1) the bank can no longer choose its disbursement policy optimally and it is restricted to c_{1h}^* even when $w_{k,\ell} = \pi$, and (2) the bank can no longer pay all withdrawers placing an order and spend all of its resources, as the bank now has to be concerned of having too many investors withdrawing to use CBDC.

Finally, we can determine the fraction of investors who use CBDC as a function of the amount of withdrawals orders,

$$w_\ell = \begin{cases} \pi \text{ \textit{(no run)}} & \text{\textit{if} } (1+r)c_{1,h}^* < \hat{c}_{2,b} \\ \in [\pi, 1] & \text{\textit{if} } (1+r)c_{1,h}^* = \hat{c}_{2,b} \\ 1 \text{ \textit{(run)}} & \text{\textit{if} } (1+r)c_{1,h}^* > \hat{c}_{2,b} \end{cases} \Rightarrow \begin{cases} \eta(\bar{w}, r) = \pi(1-\alpha_\ell)F_\ell(1+r) \\ \eta(\bar{w}, r) = \frac{\bar{w}}{w_\ell}\left[w_\ell - \pi + \pi(1-\alpha_\ell)F_\ell(1+r)\right] \vee \pi(1-\alpha_h)F_h(1+r) \\ \eta(\bar{w}, r) = \pi(1-\alpha_h)F_h(1+r) \end{cases}$$

Where $w_{k,\ell} - \pi$ is the fraction of type 2 who run on the bank, and $x \vee y$ stands for the minimum between x and y. To summarize, as we guessed, banks will serve as many withdrawers as possible, without being 'discovered' by the regulator. The optimal policy now consists in choosing τ and r to maximize investors' welfare given banks' optimal payout strategy and the withdrawal strategies of investors.

Rather than solve for the general case, we use an example to illustrate some interesting aspects of the allocation with CBDC. First, we show that CBDC can prevent a run from occurring while a run would take place without CBDC. Second, the example illustrates that with CBDC banks do not run out of resources before the bailout is put in place. Finally, with CBDC the regulator can offer a higher payment to late consumers who have not been served before the bailout, thus reducing the likelihood of a run.

5.4 Example

In this section we assume $u(c) = c^{1-\sigma}/(1-\sigma)$, where $\sigma > 1$. For most of the derivation we maintain a general form for the utility derived from consuming the public good $v(\cdot)$ but in the parametrization we assume $v(c) = c^{1-\sigma}/(1-\sigma)$.

5.4.1 Efficient allocation with $r = 0$ (no CBDC)

The efficient allocation when there is no CBDC solves the following first order conditions

$$\begin{aligned}
\left[\alpha_s u'(c_{1s}) + (1-\alpha_s)\int R_3 u'(R_3 c_{1s}) dF_s(R_3)\right] - R_s u'(c_{2s}) &= 0 \\
-v'(\tau - b_\ell) + R_\ell u'(c_{2\ell}) + \lambda_\ell &= 0 \\
(1-q)\left[v'(\tau) - R_h u'(c_{2h})\right] + q\left[v'(\tau - b_\ell) - R_\ell u'(c_{2\ell})\right] &= 0
\end{aligned}$$

together with the feasibility constraint

$$(1-\pi)c_{2s} = R_s\left[1 - \tau + b_s - \pi c_{1s}\right].$$

Define

$$\rho_s \equiv \left(\frac{R_s}{\left[\alpha_s + (1-\alpha_s)\int R_3^{1-\sigma} dF_s\left(R_3\right)\right]}\right)^{1/\sigma}$$

Then with $b_h = 0$, the efficient allocation is a vector $(c_{1s}^*, c_{2s}^*, \tau^*, b_\ell^*)$ such that

$$\begin{aligned}
c_{2h}^* &= \rho_h c_{1h}^* \\
c_{2\ell}^* &= \rho_\ell c_{1\ell}^* \\
c_{1h}^* &= \frac{1-\tau^*}{(1-\pi)\rho_h/R_h + \pi} \\
c_{1\ell}^* &= \frac{1-\tau^* + b_\ell^*}{(1-\pi)\rho_\ell/R_\ell + \pi} \\
v'(\tau^*)(1-\tau^*)^\sigma &= \frac{R_h}{\rho_h^\sigma}\left((1-\pi)\frac{\rho_h}{R_h} + \pi\right)^\sigma \\
v'(\tau^* - b_\ell^*)\left(1 - \tau^* + b_\ell^*\right)^\sigma &= \frac{R_\ell}{\rho_\ell^\sigma}\left((1-\pi)\frac{\rho_\ell}{R_\ell} + \pi\right)^\sigma
\end{aligned}$$

and $b_\ell^* > 0$ solves the last equation whenever

$$v'(\tau^*)(1-\tau^*)^\sigma > \frac{R_\ell}{\rho_\ell^\sigma}\left((1-\pi)\frac{\rho_\ell}{R_\ell} + \pi\right)^\sigma,$$

and $b_\ell^* = 0$ otherwise. In the parametrization below, we always find $b_\ell^* > 0$.

5.4.2 Run equilibrium without CBDC

Now suppose there is no CBDC and $\mathcal{X} = \{0, c_{1\ell}^*, c_{1h}^*\}$. We consider the interesting case when there is a run in the low state so that $w_\ell = 1$. We will make the appropriate parameter choice so that this is the case. As there is a run, banks extinguish their resources in the low state by paying c_{1h}^* to a measure $\bar{w}_\ell = \min\{1, (1-\tau^*)/c_{1h}^*\}$ of investors. Since $\rho_s < R_s$ (as we assumed $\sigma > 1$), banks can serve a fraction $\bar{w}_\ell = (1-\pi)\rho_h/R_h + \pi$ of investors. So banks deplete all their resources by serving c_{1h}^* to a fraction of its depositors and the remaining one will only consume thanks to the bailout. Following the bailout, the resources available at each bank is

$$\psi = \frac{\hat{b}}{1-(1-\tau^*)/c_{1h}^*} = \frac{\hat{b}}{(1-\pi)(1-\rho_h/R_h)},$$

where $\hat{b}$ solves (where we used $n_k = \pi$)

$$R_\ell u'\left(R_\ell \frac{\psi - \pi \hat{c}_{1b}}{1-\pi}\right) = v'(\tau^* - \hat{b}).$$

Using the FOC and the resource constraint, we can find $\hat{c}_{1,b}$ and $\hat{c}_{2,b}$,

$$\begin{aligned} \hat{c}_{1b} &= \frac{\hat{b}}{(1-\pi)(1-\rho_h/R_h)\left[(1-\pi)\rho_\ell/R_\ell + \pi\right]}, \\ \hat{c}_{2,b} &= \rho_\ell \hat{c}_{1b}. \end{aligned}$$

The regulator maintains the efficient consumption margin between early and late consumers after the bailout. Using the expression for the utility function, $\hat{b}$ is given by

$$v'(\tau^* - \hat{b})\hat{b}^\sigma = \frac{R_\ell}{\rho_\ell^\sigma}\left((1-\pi)\rho_\ell/R_\ell + \pi\right)^\sigma \left[(1-\pi)(1-\rho_h/R_h)\right]^\sigma \tag{7}$$

Using this $\hat{b}$ to obtain $\hat{c}_{2,b}$, we obtain that the bailout policy will yield to a run whenever $c_{1,h}^* > \hat{c}_{2,b}$. A condition for this last inequality is $R_\ell < R_h/\rho_h^\sigma$.

5.4.3 Equilibrium with CBDC

Define $\hat{\rho}_\ell(r)$ as

$$\hat{\rho}_\ell(r) \equiv \left\{ \frac{R_\ell}{\alpha_\ell + (1-\alpha_\ell)\left[\int_\rho^{1+r}(1+r)^{1-\sigma} dF_\ell(R_3) + \int_{1+r}^{\bar{\rho}} R_3^{1-\sigma} dF_\ell(R_3)\right]} \right\}^{1/\sigma}$$

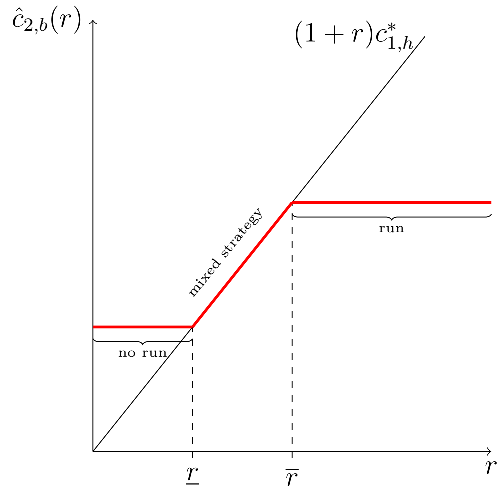

Figure 1: $\hat{c}_{2,b}(r)$

An equilibrium with CBDC is given by

$$\psi = \frac{1-\tau+b_k-\bar{w}_{k,\ell}c^*_{1,h}}{1-\bar{w}_{k,\ell}}$$
$$\hat{c}_{1,b}(r) = \frac{R_\ell\psi}{(1-n_k)\hat{\rho}_\ell(r)+R_\ell n_k}$$
$$\hat{c}_{2,b}(r) = \hat{\rho}_\ell(r)\hat{c}_{1,b}(r)$$

and

$$R_\ell u'\left(\hat{\rho}_\ell(r)\frac{R_\ell\psi}{(1-n_k)\hat{\rho}_\ell(r)+R_\ell n_k}\right) = v'\left(\tau-\int_k b_{\ell,k}dk - rc^*_{1,h}\int_k \eta(\bar{w}_{k,\ell},r)dk\right)$$

When there is a run, $w_{k,\ell}=1$ then $n(1,\bar{w}_{k,\ell})=\pi$, $\bar{w}_{k,\ell}=\frac{\pi(1-\alpha_h)F_h(1+r)}{1-\pi+\pi(1-\alpha_\ell)F_\ell(1+r)}$, and $\eta(\bar{w},r)=\pi(1-\alpha_h)F_h(1+r)$. While if there is no run $w_{k,\ell}=\pi$ then $n(\pi,\bar{w}_{k,\ell})=0$, $\bar{w}_{k,\ell}=\pi$, and $\eta(\bar{w},r)=\pi(1-\alpha_\ell)F_\ell(1+r)$. Finally, when investors of type 2 are indifferent between running and not $w_{k,\ell}\in(\pi,1)$, and $\eta(\bar{w}_{k,\ell},r)=\bar{w}_{k,\ell}\frac{(w_{k,\ell}-\pi)(1-\pi)+\pi(1-\alpha_\ell)F_\ell(1+r)}{(w_{k,\ell}-\pi)(1-\pi)+\pi}$ and $\hat{c}_{2,b}=(1+r)c^*_{1,h}$. In this latter case, the equations give the level of $w_{k,\ell}\in(\pi,1)$ consistent with this equilibrium. Figure 1 shows the allocations $\hat{c}_{2,b}(r)$ and $(1+r)c^*_{1,h}$ in the three cases.

5.4.4 Welfare

We can compute welfare with and without CBDC when the regulator only allows $\mathcal{X} = \{0, c^*_{1,\ell}, c^*_{1,h}\}$ and there is a run in the low state. The general formulation for welfare is

$$q\left\{\pi\left[\alpha_h u(c^*_{1h}) + (1-\alpha_h)u(R_3 c^*_{1h})F_h(1+r) + (1-\alpha_h)\int_{1+r}^{\bar{\rho}_h} u(R_3 c^*_{1h})dF_h\left(R_3\right)\right] + (1-\pi)u(c^*_{2h}) + v(\tau^*)\right\}$$
$$+(1-q)\bar{w}_{k,\ell}\left\{\pi\left[\alpha_\ell u(c^*_{1h}) + (1-\alpha_\ell)\int_{1+r}^{\bar{\rho}_\ell} u(R_3 c^*_{1h})dF_\ell\left(R_3\right)\right] + (1-\pi)u((1+r)c^*_{1h})\right\}$$
$$+(1-q)\left(1-\bar{w}_{k,\ell}\right)\left\{\pi\left[\alpha_\ell u(\hat{c}_{1b}) + (1-\alpha_h)\int u(R_3\hat{c}_{1b})dF_\ell\left(R_3\right)\right] + (1-\pi)u(\hat{c}_{2b})\right\}$$
$$+(1-q)v\left(\tau^* - \hat{b}(r) - rc^*_{1,h}\eta(\bar{w}_{k,\ell}, r)\right)$$

where without CBDC $\bar{w}_{k,\ell} = \frac{(1-\tau^*)}{c^*_{1h}}$ and $r = 0$. With CBDC, $\bar{w}_{k,\ell}$ and $\eta(\bar{w}_{k,\ell}, r)$ are given in section 5.4.3.

5.4.5 Illustration

To illustrate the efficient allocation, as well as the allocation post bailout without and with CBDC, we use the following parameters: $\alpha_h = 0.5 < \alpha_\ell = 0.7$ so there are more type 1 investors in the low state than in the high state. The return of the long term technology in the high and low state is $R_h = 2 > R_\ell = 1.2$, while the return on the technology of type 3 investors R_3 is distributed according to a uniform distribution with support $[1, 3]$ in state h and $[1, 2.9]$ in state ℓ. This implies that for relatively low interest rates

$$(1-\alpha_h)F_h(1+r) \geq (1-\alpha_\ell)F_\ell(1+r)$$

so that there are more investors using CBDC in the high state than in the low state when there is no run. The probability of each state occurring is $q = 0.5$. Risk aversion is $\sigma = 2$ and $\pi = 0.5$.

The following graphs shows, for different level of interest rates, the efficient allocation in the high state when stored with CBDC $(1+r)c^*_{1,h}$ (black line), the efficient allocation of type 2 investors in the low state $c^*_{2\ell}$ (orange line), the allocation of type 2 investors post bailout without CBDC $\hat{c}_{2,b}$ (red line), and the allocation post bailout of type 2 investors with CBDC $\hat{c}_{2,b}(r)$ for different level of interest rates r (blue curve).

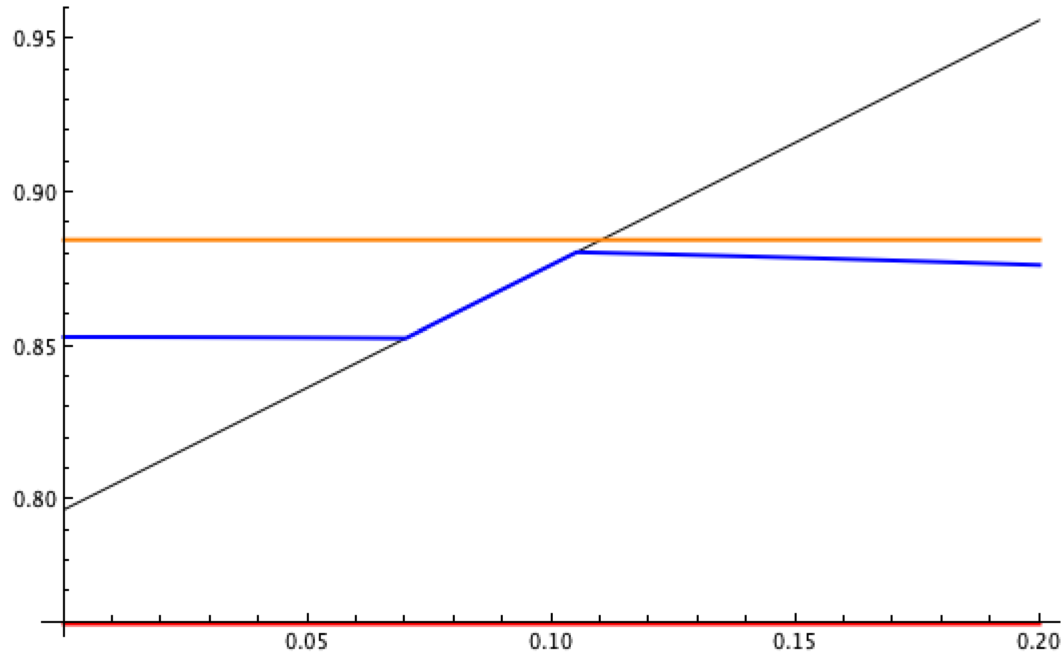

Figure 2: $(1+r)c^*_{1,h}$ (black), $c^*_{2,\ell}$ (orange), $\hat{c}_{2,b}$ (red), and $\hat{c}_{2,b}(r)$ (blue)

In this example, without CBDC, type 2 investors will always run on the bank (when the bank cheats) since $c^*_{1,h} > \hat{c}_{2,b}$. In this case the bank will spend all of its resources. The figure shows that for low rates, $\hat{c}_{2,b}(r) > (1+r)c^*_{1,h}$, while the reverse holds for high rates. So with CBDC, there is no run as long as the regulator does not pay too much on CBDC. Still the bank does cheat, and it will pay $c^*_{1,h}$ to investors who want to withdraw early. This is so because $\mathcal{X} = \{0, c^*_{1,\ell}, c^*_{1,h}\}$ and banks want to exploit the high payment to free ride on the bailout. The allocation for type 2 investors with CBDC post bailout $\hat{c}_{2,b}(r)$ is "closer" to the efficient allocation in the low state $c^*_{2,\ell}$ because the bank cannot run down all of its resources before asking for a bailout.

This example shows that the CBDC can help eliminate a run by preventing the bank from paying too much (or giving the information sooner to the regulator). However, paying too large interest on CBDC (say for monetary policy reasons that we do not model) could trigger a run. However, the regulator will be informed a run is underway before the bank runs out of resources. As a consequence, the allocation of type 2 investors with CBDC $\hat{c}_{2,b}(r)$ can be higher than without it, $\hat{c}_{2,b}$.

For completeness, let us note that increasing R_ℓ shifts the blue curve up, which makes a run less likely in an economy with CBDC.

Plotting welfare with and without CBDC as a function of the interest rate paid on CBDC,

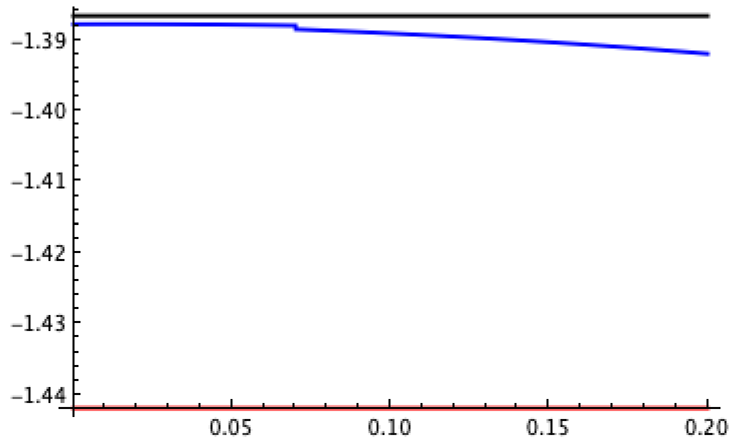

Figure 3: Welfare as a function of r, no CBDC (red), with CBDC (blue), first best with $r = 0$ (black)

we see that welfare is decreasing in the interest rate (given our utility functions, consumption equivalents are given by the ratio of these welfare levels). It may be surprising that the level of welfare with CBDC is close to the level of first best welfare when $r = 0$. The reason is this: Suppose r is arbitrarily close to zero. Then very few investors are using CBDC in the high state. As a consequence, the regulator learns the state almost instantly when the state is low. So by selecting a low r, the regulator can make welfare arbitrarily close to the first best level. Notice that there is a discontinuity at $r = 0$, since then no investor has an incentive to use CBDC.

6 Concluding Remarks

We have offered another perspective on the usefulness of providing an account based CBDC: It allows the central bank to acquire timely and correct information on the status of banks when the usual channels of communication are delayed or broken. We presented a simple example where CBDC can help the central bank in reaching a better allocation. While the details of the model are very specific, we believe the mechanism is rather general. Below, we describe how our analysis can provide indications on how to design CBDC and highlights some other aspects of the debate on CBDC.

First, it is evident that our argument does not rely on CBDC being used as a means of payment, but more as a means to protect one's savings. The central bank needs to observe that withdrawals are made from a bank and transferred into an account it manages. An account-based CBDC allows the central bank to learn the provenance of an inflow of funds (that is the owner of those funds and the owner's banks), while a token-based CBDC may hinder the acquisition of this information.

It is true that token-based CBDC maintains anonymity and depositors will lose their anonymity towards the regulators when they move their funds into a central bank account. This may hinder the adoption of an account-based CBDC. However, in times of market stress, individuals may be willing to forego some degree of anonymity for safety.

In addition to arguing for an account-based CBDC, our results show that deposit limits on these accounts, if any, should be set high enough. More precisely, the limit should be set higher than the amount that is deposited in tranquil times. Otherwise, CBDC accounts will always be fully used and the central bank will not see that investors want to rush out of the banking system. Then, in the model, a full run will occur where banks will deplete all their resources. Stepping back from the model a bit, let us consider a run when CBDC accounts would have too tight a limit such that they would always be entirely used up. In this case a run would proceed as usual: investors would withdraw from their banks and purchase safe assets. The central bank would observe an episode of flight to safety with no additional precise information on the reasons for it. How to set the limit on CDBC account is of course an important design question and it should be carefully analyzed.[10]

Another important design question is whether a CBDC should bear interest and, if so, at what rate. Our example shows the interest on CBDC should be positive when the lowest return on storage of type 3 investors is low enough ($\underline{R}_s < 1$). One may argue that this corresponds, for instance, to the case when the yield on treasuries is negative. In general, we would support the idea that the rate on CBDC be set as high as the interest rate on excess reserves or the yield on treasuries, whichever is lowest, because this would give sufficient incentives for investors to use CBDC without necessarily yielding to the disintermediation of the banking sector. Also, not paying any interest rate may drive sophisticated investors to purchase higher yielding assets. The information content in the movement of asset prices is however small relative to flows in and out of CBDC. Hence, investors should not consider deposits in CBDC as a costly investment. If the interest rate is not too low, in practice, it is likely that sophisticated investors will first park their funds into CBDC and then purchase safe assets rather than directly purchasing safe assets, thus giving the regulator the information it needs.

One may also wonder if access to (and management of) the central bank account could be outsourced to a third party. An example is a narrow bank that backs its liabilities with 100 percent central bank reserves. Our model says this would bring only some partial benefits of allowing direct access: When the narrow bank witnesses an inflow of funds, the narrow bank would act as a pass through and deposit these funds in its reserves account at the central bank.

[10] See Bindseil (2020).

The central bank would observe the size of the inflow, but again it would not be able to learn the provenance of these funds. Of course, upon the central bank's request, the narrow bank could communicate information on the funds' provenance. To the extent the implied delay is not a cause for concern, the central bank could envisage outsourcing access to its balance sheet. An additional benefit of outsourcing access to a third party is to protect the central bank against the often heard critique that holding central bank accounts may facilitate the conduct of illegal or otherwise undesirable activities.

7 References

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8 Appendix

Proof of Proposition 1

Proof. The first order condition gives us

$$h'(c_{1s}) \equiv \alpha u'(c_{1s}) + (1-\alpha) \int R_3 u'(R_3 c_{1s}) dR_3 = R_s u'(c_{2s})$$

Since we assume $\int R_3 u'(R_3 c_{1s}) dR_3 \leq u'(c_{1s})$ for all c_{1s}, we obtain $u'(c_{1s}) \geq h'(c_{1s}) = R_s u'(c_{2s}) > u'(c_{2s})$, where the last inequality follows from $R_s > 1$. Therefore, $c_{2s} > c_{1s}$ for all s. Now suppose $b_\ell > 0$. Then

$$v'(\tau - b_\ell) = R_\ell u'(c_{2\ell}) > v'(\tau) \geq R_h u'(c_{2h})$$

Therefore,

$$h'(c_{1\ell}) > h'(c_{1h})$$

by concavity of the function $h(c)$ we obtain $c_{1\ell} < c_{1h}$. Using the resource constraint, this implies

$$c_{2\ell} > \frac{R_\ell}{R_h} c_{2h}.$$

Together with $R_\ell u'(c_{2\ell}) > R_h u'(c_{2h})$, this implies $c_{2\ell} u'(c_{2\ell}) > c_{2h} u'(c_{2h})$, so that $c_{2h} > c_{2\ell}$. This shows that $c_{1h} > c_{1\ell}$ and $c_{2h} > c_{2\ell}$ whenever $b_\ell > 0$.

We now show that $b_\ell > 0$. Suppose otherwise, so that $\lambda_\ell \geq 0$ and $b_\ell = 0$. Then $v'(\tau) \geq R_\ell u'(c_{2\ell})$, and the first order condition with respect to τ together with $b_h = 0$ gives $R_h u'(c_{2h}) \geq v'(\tau)$, so that $R_h u'(c_{2h}) \geq R_\ell u'(c_{2\ell})$. The first order condition then yields $h'(c_{1h}) \geq h'(c_{1\ell})$ so that $c_{1\ell} \geq c_{1h}$. The resource constraint then implies

$$c_{2\ell} < \frac{R_\ell}{R_h} c_{2h} < c_{2h}.$$

Together with $R_h u'(c_{2h}) \geq R_\ell u'(c_{2\ell})$, this gives us $c_{2h} u'(c_{2h}) \geq c_{2\ell} u'(c_{2\ell})$ or $c_{2\ell} > c_{2h}$, a contradiction. Hence $b_\ell > 0$. This completes the proof. □

Proof of Lemma 1

Proof. The FOC with respect to $\bar{w}_\ell$ is

$$\frac{\partial V_k(w_{k,\ell}, \bar{w}_{k,\ell}, c_{1,k})}{\partial \bar{w}_{k,\ell}} = \underbrace{\left(\frac{\pi}{w_{k,\ell}}\alpha + 1 - \frac{\pi}{w_{k,\ell}}\right) u(c_{1,k}) + \frac{\pi}{w_{k,\ell}}(1-\alpha) \int u(R_3 c_{1,k}) dR_3 - V(\psi_k, n_k)}_{①} + \underbrace{(1 - \bar{w}_{k,\ell}) \frac{\partial V}{\partial n} \frac{\partial n}{\partial \bar{w}}}_{②}$$

The expression ① relates to the benefit of serving investors early on relative to serving them in a bail-out. Expression ② captures the effect on the bail-out value of serving more investors earlier on. Serving more investors earlier affects the resources of the banks, but since the allocation is independent of the bank's resources, this effect is mute. However, serving more investors earlier affects the number of type 1 and 3 investors remaining to be served after the bail-out (we would expect that paying more investors earlier on reduces the number of type 1 and 3 to be served post bail-out). The envelope conditions give

$$\frac{\partial V}{\partial n} = \alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3 - u(\hat{c}_{2,b}) \leq 0$$

the last inequality follows from the fact that $\hat{c}_{1,b} < \hat{c}_{2,b}$ and $\int_\rho^{R_h} R_3 dR_3 = 1$, so that using Jensen's inequality

$$\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3 \leq u(\hat{c}_{1,b})$$

while

$$\frac{\partial n}{\partial \bar{w}_\ell} = \frac{\pi}{w_\ell}\frac{w_\ell - 1}{(1-\bar{w}_\ell)^2} \leq 0$$

Therefore, expression (2) is positive because the number of type 1,3 investors still to be served in a bail-out is decreasing with the number of investors served prior bailout, and type 2 investors receive more in a bailout. Turning to expression (1), it is positive whenever

$$\begin{aligned} &\left(\frac{\pi}{w_{k,\ell}}\alpha + 1 - \frac{\pi}{w_{k,\ell}}\right)u(c_{1,k}) + \frac{\pi}{w_{k,\ell}}(1-\alpha)\int u(R_3 c_{1,k})dR_3 \\ &- \left[n_k\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] + (1-n_k)u(\hat{c}_{2,b})\right] > 0 \end{aligned}$$

or

$$\begin{aligned} &\left(\frac{\pi}{w_{k,\ell}}\alpha + 1 - \frac{\pi}{w_{k,\ell}}\right)u(c_{1,k}) + \frac{\pi}{w_{k,\ell}}(1-\alpha)\int u(R_3 c_{1,k})dR_3 \\ &- \left[\frac{\pi}{w_{k,\ell}}\frac{w_{k,\ell}-\bar{w}_{k,\ell}}{1-\bar{w}_{k,\ell}}\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] + (1 - \frac{\pi}{w_{k,\ell}}\frac{w_{k,\ell}-\bar{w}_{k,\ell}}{1-\bar{w}_{k,\ell}})u(\hat{c}_{2,b})\right] > 0 \end{aligned}$$

or

$$\begin{aligned} &\left(\frac{\pi}{w_{k,\ell}}\alpha + 1 - \frac{\pi}{w_{k,\ell}}\right)u(c_{1,k}) + \frac{\pi}{w_{k,\ell}}(1-\alpha)\int u(R_3 c_{1,k})dR_3 - u(\hat{c}_{2,b}) \\ &- \left[\frac{\pi}{w_{k,\ell}}\frac{w_{k,\ell}-\bar{w}_{k,\ell}}{1-\bar{w}_{k,\ell}}\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3 - u(\hat{c}_{2,b})\right]\right] > 0 \end{aligned}$$

Since $w_{k,\ell}$ is defined as

$$w_{k,\ell} = \begin{cases} \pi & \text{if } c_{1,k} < \hat{c}_{2,b} \\ \in [\pi, 1] & \text{if } c_{1,k} = \hat{c}_{2,b} \\ 1 & \text{if } c_{1,k} > \hat{c}_{2,b} \end{cases}$$

we have three cases to consider: First suppose $c_{1,k} < \hat{c}_{2,b}$ so that $w_{k,\ell} = \pi$. In this case,

$$\begin{aligned} \frac{\partial V_k(w_{k,\ell}, \bar{w}_{k,\ell}, c_{1,k})}{\partial \bar{w}_{k,\ell}} &= \alpha u(c_{1,k}) + (1-\alpha) \int u(R_3 c_{1,k}) dR_3 - u(\hat{c}_{2,b}) \\ &\quad - \left[\frac{\pi - \bar{w}_{k,\ell}}{1 - \bar{w}_{k,\ell}} \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha) \int u(R_3 \hat{c}_{1,b}) dR_3 - u(\hat{c}_{2,b}) \right] \right] \\ &\quad - \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha) \int u(R_3 \hat{c}_{1,b}) dR_3 - u(\hat{c}_{2,b}) \right] \frac{1-\pi}{(1-\bar{w}_\ell)} \\ \frac{\partial V_k(w_{k,\ell}, \bar{w}_{k,\ell}, c_{1,k})}{\partial \bar{w}_{k,\ell}} &= \alpha u(c_{1,k}) + (1-\alpha) \int u(R_3 c_{1,k}) dR_3 - \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha) \int u(R_3 \hat{c}_{1,b}) dR_3 \right] \end{aligned}$$

Suppose first $c_{1,k} < \hat{c}_{1,b}$. Then the RHS is negative, so $\bar{w}_{k,\ell} = 0$. In this case, the bank relinquishes its power to allocate resources to the planner. This cannot be an equilibrium since the bank can always choose $c_{1,k} = c^*_{1,\ell}$ and $c_{2,k} = c^*_{2,\ell}$ to achieve a higher expected payoff without resorting to a bail-out. Now suppose $c_{1,k} \geq \hat{c}_{1,b}$. Then the RHS is negative so $\bar{w}_\ell = \min\{w_\ell, (1-\tau)/c_{1,k}\}$.

Now consider the case where $c_{1,k} > \hat{c}_{2,b}$. Then $w_{k,\ell} = 1$ and

$$\begin{aligned} \frac{\partial V_k(w_{k,\ell}, \bar{w}_{k,\ell}, c_{1,k})}{\partial \bar{w}_{k,\ell}} &= \left(\frac{\pi}{w_{k,\ell}} \alpha + 1 - \frac{\pi}{w_{k,\ell}} \right) u(c_{1,k}) + \frac{\pi}{w_{k,\ell}} (1-\alpha) \int u(R_3 c_{1,k}) dR_3 - u(\hat{c}_{2,b}) \\ &\quad - \left[\frac{\pi}{w_{k,\ell}} \frac{w_{k,\ell} - \bar{w}_{k,\ell}}{1 - \bar{w}_{k,\ell}} \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha) \int u(R_3 \hat{c}_{1,b}) dR_3 - u(\hat{c}_{2,b}) \right] \right] \\ &\quad + (1 - \bar{w}_{k,\ell}) \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha) \int u(R_3 \hat{c}_{1,b}) dR_3 - u(\hat{c}_{2,b}) \right] \frac{\pi}{w_\ell} \frac{w_\ell - 1}{(1-\bar{w}_\ell)^2} \\ &= (\pi\alpha + 1 - \pi) u(c_{1,k}) + \pi(1-\alpha) \int u(R_3 c_{1,k}) dR_3 - u(\hat{c}_{2,b}) \\ &\quad - \left[\pi \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha) \int u(R_3 \hat{c}_{1,b}) dR_3 - u(\hat{c}_{2,b}) \right] \right] \\ &= \pi \left[\alpha u(c_{1,k}) + (1-\alpha) \int u(R_3 c_{1,k}) dR_3 \right] - \pi \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha) \int u(R_3 \hat{c}_{1,b}) dR_3 \right] \\ &\quad (1-\pi) \left[u(c_{1,k}) - u(\hat{c}_{2,b}) \right] \end{aligned}$$

This is positive since $c_{1,k} > \hat{c}_{2,b} > \hat{c}_{1,b}$. So when $c_{1,k} > \hat{c}_{2,b}$, then $\bar{w}_\ell = \min\{w_\ell, (1-\tau)/c_{1,k}\}$.

Finally, suppose $c_{1,k} = \hat{c}_{2,b}$ then $w_{k,\ell} \in [\pi, 1]$ and

$$\begin{aligned}
\frac{\partial V_k(w_{k,\ell}, \bar{w}_{k,\ell}, c_{1,k})}{\partial \bar{w}_{k,\ell}} &= \left(\frac{\pi}{w_{k,\ell}}\alpha + 1 - \frac{\pi}{w_{k,\ell}}\right) u(c_{1,k}) + \frac{\pi}{w_{k,\ell}}(1-\alpha)\int u(R_3 c_{1,k})dR_3 - u(\hat{c}_{2,b}) \\
&\quad - \left[\frac{\pi}{w_{k,\ell}}\frac{w_{k,\ell} - \bar{w}_{k,\ell}}{1-\bar{w}_{k,\ell}}\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3 - u(\hat{c}_{2,b})\right]\right] \\
&\quad + (1-\bar{w}_{k,\ell})\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3 - u(\hat{c}_{2,b})\right]\frac{\pi}{w_{k,\ell}}\frac{w_{k,\ell}-1}{(1-\bar{w}_{k,\ell})^2} \\
&= \frac{\pi}{w_{k,\ell}}(1-\alpha)\int u(R_3\hat{c}_{2,b})dR_3 - (1-\alpha)\frac{\pi}{w_{k,\ell}}u(\hat{c}_{2,b}) \\
&\quad - \frac{\pi}{w_{k,\ell}}\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3 - u(\hat{c}_{2,b})\right] \\
&= \frac{\pi}{w_{k,\ell}}(1-\alpha)\int \left[u(R_3\hat{c}_{2,b}) - u(R_3\hat{c}_{1,b})\right]dR_3 + \alpha\frac{\pi}{w_{k,\ell}}\left[u(\hat{c}_{2,b}) - u(\hat{c}_{1,b})\right]
\end{aligned}$$

Since $\hat{c}_{2,b} > \hat{c}_{1,b}$, the RHS is positive so that $\bar{w}_\ell = \min\{w_\ell, (1-\tau)/c_{1,k}\}$. □

Proof of Lemma 2

Proof. Suppose $w_{k,\ell} > \pi$. Then

$$\begin{aligned}
V_k(w_{k,\ell}, \bar{w}_{k,\ell}, c_{1,k}) &= \bar{w}_{k,\ell}\left[\left(\tfrac{\pi}{w_{k,\ell}}\alpha + 1 - \tfrac{\pi}{w_{k,\ell}}\right)u(c_{1,k}) + \tfrac{\pi}{w_{k,\ell}}(1-\alpha)\textstyle\int u(R_3 c_{1,k})dR_3\right] \\
&\quad + (1-\bar{w}_{k,\ell})\left[n_k\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] + (1-n_k)u(\hat{c}_{2,b}))\right]
\end{aligned}$$

together with

$$n_k = \frac{\pi}{w_{k,\ell}}\frac{w_{k,\ell} - \bar{w}_{k,\ell}}{1-\bar{w}_{k,\ell}}$$

we obtain

$$
\begin{aligned}
V_k(w_{k,\ell}, \bar{w}_{k,\ell}, c_{1,k}) &= \bar{w}_{k,\ell}\left[\left(\tfrac{\pi}{w_{k,\ell}}\alpha + 1 - \tfrac{\pi}{w_{k,\ell}}\right)u(c_{1,k}) + \tfrac{\pi}{w_{k,\ell}}(1-\alpha)\int u(R_3 c_{1,k})dR_3\right] \\
&\quad +(1-\bar{w}_{k,\ell})\left[\begin{array}{r} \frac{\pi}{w_{k,\ell}}\frac{w_{k,\ell}-\bar{w}_{k,\ell}}{1-\bar{w}_{k,\ell}}\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] \\ +(1-\frac{\pi}{w_{k,\ell}}\frac{w_{k,\ell}-\bar{w}_{k,\ell}}{1-\bar{w}_{k,\ell}})u(\hat{c}_{2,b})) \end{array}\right] \\
&= \bar{w}_{k,\ell}\left[\left(\tfrac{\pi}{w_{k,\ell}}\alpha + 1 - \tfrac{\pi}{w_{k,\ell}}\right)u(c_{1,k}) + \tfrac{\pi}{w_{k,\ell}}(1-\alpha)\int u(R_3 c_{1,k})dR_3\right] \\
&\quad + \tfrac{\pi}{w_{k,\ell}}(w_{k,\ell}-\bar{w}_{k,\ell})\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] \\
&\quad + \left[1-\bar{w}_{k,\ell} - \frac{\pi}{w_{k,\ell}}(w_{k,\ell}-\bar{w}_{k,\ell})\right]u(\hat{c}_{2,b})) \\
&= \bar{w}_{k,\ell}\left[\left(\tfrac{\pi}{w_{k,\ell}}\alpha + 1 - \tfrac{\pi}{w_{k,\ell}}\right)u(c_{1,k}) + \tfrac{\pi}{w_{k,\ell}}(1-\alpha)\int u(R_3 c_{1,k})dR_3\right] \\
&\quad + \left(\pi - \frac{\pi}{w_{k,\ell}}\bar{w}_{k,\ell}\right)\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] \\
&\quad + \left[1-\pi-\bar{w}_{k,\ell}\left(1-\frac{\pi}{w_{k,\ell}}\right)\right]u(\hat{c}_{2,b})) \\
&= \bar{w}_{k,\ell}\left[\begin{array}{c} \frac{\pi}{w_{k,\ell}}\alpha\left[u(c_{1,k}) - u(\hat{c}_{1,b})\right] + \left(1-\frac{\pi}{w_{k,\ell}}\right)\left[u(c_{1,k}) - u(\hat{c}_{2,b})\right] \\ +\frac{\pi}{w_{k,\ell}}(1-\alpha)\int\left[u(R_3 c_{1,k}) - u(R_3\hat{c}_{1,b})\right]dR_3 \end{array}\right] \\
&\quad +\pi\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] + (1-\pi)u(\hat{c}_{2,b})
\end{aligned}
$$

Since $w_{k,\ell} > \pi$ the ratio $\pi/w_{k,\ell} < 1$, so the first line is increasing in $c_{1,k}$ (direct effect). Also, $w_{k,\ell} > \pi$ if $c_{1,k} > \hat{c}_{2,b} > \hat{c}_{1,b}$. Since $\bar{w}_{k,\ell} = \min\{w_{k,\ell}, (1-\tau)/c_{1,k}\}$, the optimal policy may not be $c_{1,k} = \infty$ (or whatever the highest $c \in \mathcal{X}$ is). Indeed, the FOC when $c_{1,k}$ becomes large (so that $\bar{w}_{k,\ell} = (1-\tau)/c_{1,k}$ holds) gives

$$
\begin{aligned}
&\bar{w}_{k,\ell}\left[\frac{\pi}{w_{k,\ell}}\alpha u'(c_{1,k}) + \left(1-\frac{\pi}{w_{k,\ell}}\right)u'(c_{1,k}) + \frac{\pi}{w_{k,\ell}}(1-\alpha)\int R_3 u'(R_3 c_{1,k})dR_3\right] + \\
&\frac{-(1-\tau)}{(c_{1,k})^2}\left[\begin{array}{c} \frac{\pi}{w_{k,\ell}}\alpha\left[u(c_{1,k}) - u(\hat{c}_{1,b})\right] + \\ \left(1-\frac{\pi}{w_{k,\ell}}\right)\left[u(c_{1,k}) - u(\hat{c}_{2,b})\right] + \frac{\pi}{w_{k,\ell}}(1-\alpha)\int\left[u(R_3 c_{1,k}) - u(R_3\hat{c}_{1,b})\right]dR_3 \end{array}\right] = 0
\end{aligned}
$$

This expression can also be written as

$$\bar{w}_{k,\ell}\left\{\begin{array}{c}\frac{\pi}{w_{k,\ell}}\alpha\left[u'(c_{1,k})-\frac{u(c_{1,k})}{c_{1,k}}\right]+\left(1-\frac{\pi}{w_{k,\ell}}\right)\left[u'(c_{1,k})-\frac{u(c_{1,k})}{c_{1,k}}\right]\\+\frac{\pi}{w_{k,\ell}}(1-\alpha)\int R_3\left[u'(R_3c_{1,k})-\frac{u(R_3c_{1,k})}{c_{1,k}}\right]dR_3\end{array}\right\}+$$
$$+\frac{\bar{w}_{k,\ell}}{c_{1,k}}\left[\frac{\pi}{w_{k,\ell}}\alpha u(\hat{c}_{1,b})+\left(1-\frac{\pi}{w_{k,\ell}}\right)u(\hat{c}_{2,b})+\frac{\pi}{w_{k,\ell}}(1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] = 0$$

Hence, dividing by $\bar{w}_{k,\ell}$, we obtain

$$\begin{aligned}\frac{\pi}{w_{k,\ell}}\alpha\left[u'(c_{1,k})-\frac{u(c_{1,k})}{c_{1,k}}\right]+\left(1-\frac{\pi}{w_{k,\ell}}\right)\left[u'(c_{1,k})-\frac{u(c_{1,k})}{c_{1,k}}\right]+&\\ \frac{\pi}{w_{k,\ell}}(1-\alpha)\int R_3\left[u'(R_3c_{1,k})-\frac{u(R_3c_{1,k})}{c_{1,k}}\right]dR_3+&\\ \frac{1}{c_{1,k}}\left[\frac{\pi}{w_{k,\ell}}\alpha u(\hat{c}_{1,b})+\left(1-\frac{\pi}{w_{k,\ell}}\right)u(\hat{c}_{2,b})+\frac{\pi}{w_{k,\ell}}(1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] &= 0\end{aligned}$$

The last term is always positive and goes to zero as $c_{1,k}\to\infty$. Since u is concave, $u'(c)<u(c)/c$ for all c, which implies the first and second terms are always negative and will eventually dominate the last term. It follows that there exists $\tilde{c}_{1,k}$ such that

$$\frac{\partial V_k(w_{k,\ell},\frac{(1-\tau)}{\tilde{c}_{1,k}},\tilde{c}_{1,k})}{\partial c_{1,k}}=0$$

Finally, suppose the solution has $\bar{w}_{\ell,k}=w_{\ell,k}$ so that the bank does not extinguish its resources. Then the FOC are always positive, so that the bank wants to increase $c_{1,k}$. This shows that $\bar{w}_{\ell,k}=(1-\tau)/\tilde{c}_{1,k}$. □

Proof of Lemma 3:

Proof. Assume $\pi < (1-\tau)/\hat{c}_{2,b}$. When $w_{k,\ell} = \pi$, we know from (4) that $c_{1,k} \leq \hat{c}_{2,b}$. Then

$$\begin{aligned}
V_k(w_{k,\ell}, \bar{w}_{k,\ell}, c_{1,k}) &= \bar{w}_{k,\ell}\left[\alpha u(c_{1,k}) + (1-\alpha)\int u(R_3 c_{1,k})dR_3\right] \\
&\quad + (1-\bar{w}_{k,\ell})\left[n_k\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] + (1-n_k)u(\hat{c}_{2,b}))\right] \\
&= \bar{w}_{k,\ell}\left[\alpha u(c_{1,k}) + (1-\alpha)\int u(R_3 c_{1,k})dR_3\right] \\
&\quad + \frac{\pi}{w_{k,\ell}}\frac{(1-\bar{w}_{k,\ell})(w_{k,\ell}-\bar{w}_{k,\ell})}{1-\bar{w}_{k,\ell}}\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] \\
&\quad + \left[(1-\bar{w}_{k,\ell}) - \frac{\pi}{w_{k,\ell}}(1-\bar{w}_{k,\ell})\frac{w_{k,\ell}-\bar{w}_{k,\ell}}{1-\bar{w}_{k,\ell}}\right]u(\hat{c}_{2,b})) \\
&= \bar{w}_{k,\ell}\left[\alpha u(c_{1,k}) + (1-\alpha)\int u(R_3 c_{1,k})dR_3\right] \\
&\quad + \left[(\pi-\bar{w}_{k,\ell})\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] + (1-\pi)u(\hat{c}_{2,b}))\right] \\
&= \bar{w}_{k,\ell}\left\{\left[\alpha u(c_{1,k}) + (1-\alpha)\int u(R_3 c_{1,k})dR_3\right] - \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right]\right\} \\
&\quad + \left[\pi\left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right] + (1-\pi)\,u(\hat{c}_{2,b}))\right]
\end{aligned}$$

and its derivative with respect to $c_{1,k}$ is

$$\begin{aligned}
&\bar{w}_{k,\ell}\left[\alpha u'(c_{1,k}) + (1-\alpha)\int R_3 u'(R_3 c_{1,k})dR_3\right] \\
&+ \left\{\left[\alpha u(c_{1,k}) + (1-\alpha)\int u(R_3 c_{1,k})dR_3\right] - \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right]\right\}\frac{\partial \bar{w}_{k,\ell}}{\partial c_{1,k}}
\end{aligned}$$

Since $\pi < (1-\tau)/\hat{c}_{2,b}$ then $\bar{w}_{k,\ell} = \pi$ (and $\frac{\partial \bar{w}_{k,\ell}}{\partial c_{1,k}} = 0$) so the expression above is always positive and so the constraint $c_{1,k} \leq \hat{c}_{2,b}$ binds. For completeness, suppose $(1-\tau)/\hat{c}_{1,b} > \pi > (1-\tau)/\hat{c}_{2,b}$. Then $c_{1,b} > \hat{c}_{1,b}$ and the expression in $\{.\}$ is positive. The bank will increase $c_{1,k}$ until the resource constraint binds, so that $\bar{w}_{k,\ell} = (1-\tau)/c_{1,k}$. So the derivative becomes

$$\begin{aligned}
&\frac{1-\tau}{(c_{1,k})}\left[\alpha u'(c_{1,k}) + (1-\alpha)\int R_3 u'(R_3 c_{1,k})dR_3\right] \\
&- \frac{1-\tau}{(c_{1,k})^2}\left\{\left[\alpha u(c_{1,k}) + (1-\alpha)\int u(R_3 c_{1,k})dR_3\right] - \left[\alpha u(\hat{c}_{1,b}) + (1-\alpha)\int u(R_3\hat{c}_{1,b})dR_3\right]\right\}
\end{aligned}$$

Since the second term is negative, it could well be that the bank chooses to reduce the payment $c_{1,k}$ possibly below $\hat{c}_{2,b}$. However, our assumption that $\pi < (1-\tau)/\hat{c}_{2,b}$ implies we are never in this case. ☐

Proof of Lemma 4

Proof. Since $r = 0$, the regulator does not make a payment to holders of CBDC and $\gamma_s(0) = 0$. Then the first order conditions are

$$\begin{aligned} z_s'(c_{1s}) - R_s u'(c_{2s}) &= 0 \\ -v'(\tau - b_\ell) + R_\ell u'(c_{2\ell}) + \lambda_\ell &= 0 \\ (1-q)\left[v'(\tau) - R_h u'(c_{2h})\right] + q\left[v'(\tau - b_\ell) - R_\ell u'(c_{2\ell})\right] &= 0 \end{aligned}$$

where

$$\tilde{z}_s'(c_{1s}) \equiv \alpha_s u'(c_{1s}) + (1-\alpha_s)\int \tilde{R}_3 u'(\tilde{R}_3 c_{1s}) f_s(R_3) dR_3.$$

We have to show that $c^*_{1h} > c^*_{1\ell}$. There is one difficulty: $\tilde{z}_s$ is a function of the state. First notice that

$$\tilde{z}_s'(c_{1s}) = R_s u'(c_{2s}) \tag{8}$$

so that using our assumptions we obtain $c_{1s} < c_{2s}$. Now suppose $b_\ell > 0$. Then

$$v'(\tau - b_\ell) = R_\ell u'(c_{2\ell}),$$

and

$$v'(\tau - b_\ell) = R_\ell u'(c_{2\ell}) > v'(\tau) \geq R_h u'(c_{2h}),$$

so that using (8), we obtain $\tilde{z}_\ell'(c_{1\ell}) > \tilde{z}_h'(c_{1h})$. Since $\tilde{z}_\ell'(c) \leq \tilde{z}_h'(c)$ and by concavity of the function $\tilde{z}_s$ it must be that $c_{1\ell} < c_{1h}$. Using the resource constraint

$$c_{2\ell} \;>\; \frac{R_\ell}{R_h} c_{2h}.$$

Together with $R_\ell u'(c_{2\ell}) > R_h u'(c_{2h})$, this implies $c_{2\ell} u'(c_{2\ell}) > c_{2h} u'(c_{2h})$, so that $c_{2h} > c_{2\ell}$. Therefore $(c_{1h}, c_{2h}) > (c_{1\ell}, c_{2\ell})$ whenever $b_\ell > 0$.

Now suppose $b_\ell = 0$ then

$$v'(\tau) \geq R_\ell u'(c_{2\ell})$$

but the FOC with respect to τ gives us $R_h u'(c_{2h}) \geq R_\ell u'(c_{2\ell})$. So the first order condition then yields $z_h'(c_{1h}) \geq z_\ell'(c_{1\ell})$. Suppose $c_{1\ell} \geq c_{1h}$ then the resource constraint implies $c_{2\ell} = R_\ell [1-\tau-\pi c_{1\ell}] \leq R_h [1-\tau-\pi c_{1h}] = c_{2h}$. Also

$$c_{2\ell} < \frac{R_\ell}{R_h} c_{2h} < c_{2h}.$$

So together with $R_h u'(c_{2h}) \geq R_\ell u'(c_{2\ell})$, this gives us $c_{2h} u'(c_{2h}) \geq c_{2\ell} u'(c_{2\ell})$ or $c_{2\ell} > c_{2h}$, a contradiction. So it must be that $c_{1h} \geq c_{1\ell}$ (even when $b_\ell = 0$ – which here can be a possibility.) □

$$\max \quad \sum_{s=h,\ell} q_s \left\{ \pi \left[\alpha_s u(c_{1s}) + (1-\alpha_s) \int_\rho^{1+r} u((1+r)c_{1s}) f_s(R_3) dR_3 + (1-\alpha_s) \int_{1+r}^{\bar\rho} u(R_3 c_{1s}) f_s(R_3) dR_3 \right] + (1-\pi) u(c_{2s}) + v\left(\tau - b_s - \underbrace{\pi(1-\alpha_s) r F_s(1+r) c_{1s}}_{C_s(r)} \right) \right\}$$

subject to

$$\begin{aligned} (1-\pi)c_{2s} &= R_s [1-\tau+b_s-\pi c_{1s}] \\ b_\ell &\geq 0 \quad \text{and } b_h = 0 \end{aligned}$$

FOC with respect to r:

$$\sum_{s=h,\ell} q_s \pi (1-\alpha_s) \left\{ \underbrace{c_{1s} u'((1+r)c_{1s})}_{\text{MU higher rates}} \underbrace{F_s(1+r)}_{\#\text{ investors benefiting}} - \underbrace{[F_s(1+r)c_{1s} + r f_s(1+r) c_{1s}]}_{\text{reduction in public good}} \underbrace{[R_s u'(c_{2s}) + \lambda_s]}_{\text{MU public consumption}} \right\} = 0 \qquad (9)$$

So the expected marginal value of the better techno for type 3 investors has to equal the expected marginal value of public consumption weighted by the the loss of public consumption due to higher interest payments. So when the technology of investors 3 is bad, we would expect that it is optimal to pay interests on CDBC.

The FOC with respect to c_{1s}, b_ℓ, and τ are respectively,

$$\begin{aligned} \alpha_s u'(c_{1s}) + (1-\alpha_s)(1+r) u'((1+r)c_{1s}) F_s(1+r) + (1-\alpha_s) \int_{1+r}^{\bar\rho} R_3 u'(R_3 c_{1s}) f_s(R_3) dR_3 - R_s u'(c_{2s}) &= 0 \\ -v'(\tau - b_\ell - C_\ell(r)) + R_\ell u'(c_{2\ell}) + \lambda_\ell &= 0 \\ (1-q)\left[v'\left(\tau - C_h(r)\right) - R_h u'(c_{2h})\right] + q\left[v'\left(\tau - b_\ell - C_\ell(r)\right) - R_\ell u'(c_{2\ell})\right] &= 0 \end{aligned}$$

At $1+r \approx \bar{\rho} > 1$ we have

$$\alpha_s u'(c_{1s}) + (1-\alpha_s)\bar{\rho}u'(\bar{\rho}c_{1s}) \quad = \quad R_s u'(c_{2s}) > u'(\bar{\rho}c_{1s})$$

and the first order condition with respect to r gives

$$\sum_{s=h,\ell} q_s \pi(1-\alpha_s)F_s(\bar{\rho}) \left\{ c_{1s}u'(\bar{\rho}c_{1s}) - \left[c_{1s} + r\frac{f_s(\bar{\rho})}{F_s(\bar{\rho})}c_{1s} \right] [R_s u'(c_{2s}) + \lambda_s] \right\} < 0$$

Therefore, the regulator wants to set $1+r < \bar{\rho}$.

Now suppose the lower bound for R_s is $\rho_s = 1$ then (9) holds is zero at $r=0$ since $F_s(1) = 0$.

Suppose now the lower bound for R_ℓ is $\rho_\ell < 1$ and the one for R_h is $\rho_h = 1$, then (9) evaluated at $r = 0$,

$$q_\ell \pi(1-\alpha_\ell) \{c_{1\ell}u'(c_{1\ell})F_\ell(1) - F_\ell(1)c_{1\ell} [R_\ell u'(c_{2\ell}) + \lambda_\ell]\}$$

which is proportional to $u'(c_{1\ell}) - [R_\ell u'(c_{2\ell}) + \lambda_\ell]$. The FOC with respect to $c_{1\ell}$ implies

$$\alpha_\ell u'(c_{1\ell}) + (1-\alpha_\ell)u'(c_{1\ell})F_\ell(1) + (1-\alpha_\ell)\int_1^{\bar{\rho}} R_3 u'(R_3 c_{1\ell}) f_\ell(R_3) dR_3 = R_\ell u'(c_{2\ell})$$

and since $cu'(c)$ is decreasing in c, $R_\ell u'(c_{2\ell}) < u'(c_{1\ell})$.

Suppose now the lower bound for R_h is $\rho_h < 1$ and the one for R_ℓ is $\rho_\ell = 1$, then (9) evaluated at $r = 0$,

$$q_h \pi(1-\alpha_h) \{c_{1h}u'(c_{1h})F_h(1) - F_h(1)c_{1h}R_h u'(c_{2h})\}$$

which is proportional to $u'(c_{1h}) - R_h u'(c_{2h})$. The FOC with respect to $c_{1\ell}$ implies

$$\alpha_h u'(c_{1h}) + (1-\alpha_h)u'(c_{1h})F_h(1) + (1-\alpha_h)\int_1^{\bar{\rho}} R_3 u'(R_3 c_{1h}) f_h(R_3) dR_3 = R_h u'(c_{2\ell})$$

and since $cu'(c)$ is decreasing in c, $R_h u'(c_{2h}) < u'(c_{1h})$. Therefore the LHS of (9) is positive when evaluated at $r = 0$ and the optimal r is positive.

Suppose $\rho_h = \rho_\ell = \rho$ then at $1+r \approx \rho$ we have

$$\alpha_s u'(c_{1s}) + (1-\alpha_s)\int_\rho^{\bar{\rho}} R_3 u'(R_3 c_{1s}) f_s(R_3) dR_3 \quad = \quad R_s u'(c_{2s}) < u'(\rho c_{1s})$$

and the first order condition with respect to r gives (since $\lambda_s = 0$ for all s)

$$\begin{aligned}
\sum_{s=h,\ell} q_s\pi(1-\alpha_s)F_s(\rho)\left\{c_{1s}u'(\rho c_{1s}) - \left[c_{1s} + (\rho-1)\frac{f_s(\rho)}{F_s(\rho)}c_{1s}\right]R_s u'(c_{2s})\right\} &> \\
\sum_{s=h,\ell} q_s\pi(1-\alpha_s)F_s(\rho)\left\{c_{1s}u'(\rho c_{1s}) - \left[c_{1s} + (\rho-1)\frac{f_s(\rho)}{F_s(\rho)}c_{1s}\right]u'(\rho c_{1s})\right\} &= \\
\sum_{s=h,\ell} q_s\pi(1-\alpha_s)F_s(\rho)\left\{(1-\rho)\frac{f_s(\rho)}{F_s(\rho)}c_{1s}u'(\rho c_{1s})\right\}
\end{aligned}$$

which is positive if $1 \geq \rho$. In this case, the regulator will want to set $1 + r > \rho$. By continuity there is $\hat{\rho}$ such that the regulator wants to set $r > 0$ for $\rho < \hat{\rho}$ and zero above.

8.1 Example

Suppose $u(c) = c^{1-\sigma}/(1-\sigma)$ and CBDC pays rate r. Then the optimal allocation $(c^*_{1s}(r), c^*_{2s}(r))$ solve

$$\begin{aligned}
\alpha_s u'(c_{1s}) + (1-\alpha_s)(1+r)u'((1+r)c_{1s})F_s(1+r) + (1-\alpha_s)\int_{1+r}^{\bar{\rho}} R_3 u'(R_3 c_{1s})f_s(R_3)dR_3 - R_s u'(c_{2s}) &= 0 \\
-v'(\tau - b_\ell - C_\ell(r)) + R_\ell u'(c_{2\ell}) + \lambda_\ell &= 0 \\
(1-q)\left[v'\left(\tau - C_h(r)\right) - R_h u'(c_{2h})\right] + q\left[v'\left(\tau - b_\ell - C_\ell(r)\right) - R_\ell u'(c_{2\ell})\right] &= 0
\end{aligned}$$

and using $v(g) = Ag^{1-\sigma}/(1-\sigma)$, the feasibility constraint, and some algebra, we obtain

$$\begin{aligned}
c_{2s} &= \rho_s(r)c_{1s} \\
c_{1\ell} &= \frac{1}{\frac{(1-\pi)\rho_\ell(r)}{R_\ell} + \pi + \pi(1-\alpha_\ell)rF_\ell(1+r) + \left(\frac{R_\ell}{A}\right)^{-1/\sigma}\rho_\ell(r)} \\
c_{1h} &= \frac{1}{\frac{(1-\pi)}{R_h}\rho_h(r) + \pi + \pi(1-\alpha_h)rF_h(1+r) + \left(\frac{R_h}{A}\right)^{-1/\sigma}\rho_h(r)} \\
b_\ell &= \frac{(1-\pi)c_{2\ell}}{R_\ell} - (1 - \tau - \pi c_{1\ell}) \\
\tau &= 1 - \pi c_{1h} - \frac{(1-\pi)c_{2h}}{R_h}
\end{aligned}$$

For the same parameters as before, but with $\underline{R}_h = \underline{R}_\ell = 0.9$, welfare as a function of r is shown in Figure 3. It is clear that in this case, $r^* > 0$. For $\underline{R}_h = \underline{R}_\ell = 1$, welfare is maximized for $r = 0$.

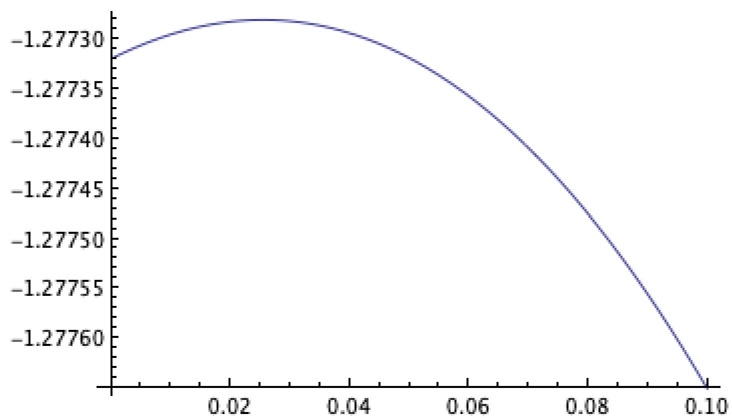

Figure 4: Welfare with CBDC as a function of r