Central Bank Digital Currency and Monetary Policy

Seyed Mohammadreza Davoodalhosseini*

May 20, 2020

Abstract

Many central banks are contemplating whether to issue a central bank digital currency (CBDC). A CBDC has certain potential benefits, including the possibility that it can bear interest. However, using a CBDC is costly for agents, perhaps because they lose their anonymity when using the CBDC instead of cash. I study the optimal monetary policy when only cash, only a CBDC, or both cash and a CBDC are available to agents. If the cost of using a CBDC is not too high, more efficient allocations can be implemented by using a CBDC than with cash, and the first best can be achieved. Having both cash and a CBDC available may result in lower welfare than in the cases where only cash or only a CBDC is available. The welfare gains of introducing a CBDC are estimated under various scenarios for the US and Canada. For example, if the monetary cost of using CBDC relative to cash is around 0.25% of the transaction value, introducing a CBDC will lead to an increase of 0.12%-0.21% consumption for the US and 0.04%-0.07% for Canada.

Keywords: Central bank digital currency, cash, monetary policy.

JEL: E42, E50.

*Bank of Canada, davo@bankofcanada.ca. I would like to thank Jonathan Chiu, Charles Kahn and Francisco Rivadeneyra for their helpful comments and suggestions. I would like also to thank Marco Bassetto, Wilko Bolt, Pedro Gomis-Porqueras, Scott Henry, Tai-Wei Hu, Janet Jiang, Todd Keister, Sephorah Mangin, Venky Venkateswaran, Steve Williamson, Cathy Zhang, Yu Zhu, and the participants of the seminars at the Bank of Canada; Monash University; Central Bank Research Association; the Summer Workshop on Money, Banking, Payments and Finance; and the Workshop of the Australasian Macroeconomics Society. The views expressed in this paper are solely those of the author and no responsibility for them should be attributed to the Bank of Canada.

"[P]hasing out paper currency is arguably the simplest and most elegant approach to clearing the path for central banks to invoke unfettered negative interest rate policies should they bump up against the 'zero lower bound' on interest rates."

Ken Rogoff (2016) in *The Curse of Cash*

"Some economists advocate that the central bank should replace cash with a digital currency that can be given a negative interest rate. ... I would once again like to say that the Riksbank has a statutory requirement to issue banknotes and coins. I see e-krona primarily as a complement to cash."

Skingsley (2016), Deputy Governor of the Bank of Sweden

1 Introduction

There has been a great deal of discussion in recent years about the potential effects of introducing a central bank digital currency (CBDC) into economies and whether cash should be eliminated, as the quotes above indicate. Some central banks have already started the decision-making process on whether to introduce a CBDC into their respective economies. For example, the Sweden's Riksbank has already started testing a CBDC (what they call "e-krona") to show how general public could use it.[1] Some officials at the central bank of China have expressed their desire to issue their own digital currency as a way to support their digital economy.[2] If central banks issue a CBDC, important questions arise, some of which are as follows: Should central banks eliminate cash from circulation? What would be the optimal (i.e., welfare-maximizing) monetary policy if agents can choose between cash and a CBDC? And quantitatively, what would be the welfare gains of introducing a CBDC into the economy?

To address these and similar questions, I use the framework of Lagos and Wright (2005) to build a model in which two means of payment could be available to agents: cash and a CBDC. What I mean by a CBDC in this paper is the money issued by the central bank in electronic format and universally accessible; i.e., all agents in the economy can use it

[1] See two reports by the Riksbank on E-krona Project (Sveriges Riksbank (2017) and Sveriges Riksbank (2018)). Also note that a CBDC can be of different types. Several authors have suggested taxonomies to understand various types of electronic money, some forms of which are a CBDC. See Bech and Garratt (2017), Bjerg (2017) or the Committee on Payments and Market Infrastructures (2015) report on digital currencies.

[2] See here: https://goo.gl/kEpHhV.

to purchase goods and services.[3] I study the optimal monetary policy when only one or both means of payment are available to agents. Cash and a CBDC are different along two dimensions in this paper. First, the ability of the central bank to implement monetary policy is different across these means of payment. The central bank can allocate transfers to agents based on their CBDC balances but cannot do so based on their cash balances because the central bank cannot see agents' cash balances. Therefore, the only policy that the central bank can implement with cash is to distribute the newly created cash evenly across all agents or through an Open Market Operation (OMO, exchanging cash by CBDC).[4] Second, carrying a CBDC is more costly for agents relative to cash. This cost is perhaps due to the fact that agents lose their anonymity if using a CBDC. This cost creates a sensible tradeoff for the central bank regarding the means of payment that the central bank would like agents to use. While a CBDC is a more flexible policy instrument, it is more costly for agents than cash.

There are two main results of the paper. First, given that the cost of carrying a CBDC is not too high, the fact that the CBDC is interest bearing in a non-linear fashion allows the central bank to achieve better allocations than with cash. In particular, it is possible to achieve the first-best level of production by using the CBDC if the agents are patient enough and if the bargaining power of buyers is sufficiently high, while it is never possible to achieve the first best by using cash. Second, when cash and a CBDC are both available to agents and valued in equilibrium, the monetary policy may be more constrained (i.e., welfare may be lower) compared with the case in which only one means of payment is available.

To elaborate on these results, consider three different schemes: only cash is available to the agents (cash-only scheme), only a CBDC is available to the agents (CBDC-only scheme), and both cash and a CBDC are available (co-existence scheme).[5] If only cash is available, then the optimal inflation in this economy is zero. A negative inflation rate would be impossible to be implemented, as the central bank cannot force agents to pay taxes on their cash balances, and a positive inflation rate would lead agents to economize on their real balances relative to the first best, so the production level would be distorted. If only a

[3] A similar form of money was proposed by Tobin (1987). In his own words: "I think the government should make available to the public a medium with the convenience of deposits and the safety of currency, essentially currency on deposit, transferable in any amount by check or other order."

[4] Note that taxing cash balances is assumed not to be feasible here; otherwise, the central bank can simply run the Friedman rule to achieve the first best, and adding a CBDC or replacing cash by a CBDC would not offer any potential improvements.

[5] The optimal monetary policy under co-existence is defined to be one that maximizes welfare among all policies under which both cash and CBDC are valued in equilibrium and used as means of payment.

CBDC is available, then the set of implementable allocations is larger, because the balance-contingent transfers are allowed with the CBDC, not with cash, and even the first-best level of production can be achieved. However, there is welfare loss resulting from the cost of carrying the CBDC. Comparing cash-only and CBDC-only schemes, I find that the tradeoff for the central bank is simply between distorting the allocation relative to the first best under the cash-only scheme or having the agents incur the cost of carrying the CBDC under the CBDC-only scheme.

Under the co-existence scheme, agents with lower transaction needs endogenously choose to use cash and agents with higher transaction needs choose to use the CBDC. In this case, the central bank faces a constraint stemming from the endogenous choice of means of payment. Because cash is available, agents whose welfare level is higher under the CBDC-only scheme relative to the cash-only scheme can now use cash as a way to evade the taxation that CBDC users are subjected to. To discourage these agents from using the CBDC, the central bank could set the cash inflation too high, but it would hurt cash users. Therefore, the availability of cash in the presence of a CBDC imposes a constraint for the central bank's maximization problem. Whether or not the co-existence scheme is optimal (i.e., leading to higher welfare) relative to cash-only or CBDC-only schemes depends on how tight this constraint is. If the constraint is too tight, then the central bank would prefer to have only one means of payment used by agents. In this case, if the cost of carrying the CBDC is not too high, then central bank eliminates cash, and if the cost is too high, then the central bank eliminates the CBDC. On the other hand, if the constraint is relatively relaxed, then the central bank would have both cash and CBDC circulate in the economy.[6]

For both cash and a CBDC to be used by agents, the cash inflation must be strictly positive. This result is obtained despite the fact that, when both cash and a CBDC are available, it may seem feasible to implement a negative cash inflation rate through an OMO. However, a negative inflation rate on cash would induce users to switch from the CBDC to cash, as the return on holding cash would be higher than holding the CBDC and agents do not need to incur the utility cost of carrying the CBDC. Since the CBDC would not be used, the central bank could not conduct an OMO under a negative cash inflation rate.

Rogoff (2016) has argued in favor of eliminating cash from circulation, except perhaps for small-denomination notes, as the first quote above indicates. One of his main arguments is that by eliminating cash, central banks can stimulate the economy in downturns via setting negative nominal interest rates. If cash is available, since cash guarantees the nominal interest

[6]In my quantitative exercise, the co-existence scheme is unlikely to be optimal for the US and can be optimal for Canada only if the cost of a CBDC is within a small range of values.

rate of zero for agents, the ability of central banks to stimulate the economy will be restricted. The reason that co-existence of cash and a CBDC may not be optimal in my model is similar to Rogoff's argument. In both, cash provides an outside option for agents, restricting the set of feasible allocations that the central bank can achieve. However, it is important to note that the effectiveness of a CBDC is not only due to the fact that it allows for the possibility of achieving negative interest rates, but it also allows for the implementation of non-linear transfer schemes, the feature that I use to show that the first-best level of production can be achieved using CBDC.[7] Altogether, even if cash is not eliminated, a CBDC can still positively affect the monetary policy, although its effectiveness is sometimes enhanced if cash is eliminated.[8]

To give a sense of the welfare gains of introducing a CBDC, I calibrate the model to the Canadian and US data. I show that introducing a CBDC can lead to an increase of up to 0.15% in consumption for Canada and up to 0.34% for the US, compared with their respective economies if only cash is used. Assuming that there are only two sizes of transactions (large-value and small-value transactions), I calculate the welfare gains of introducing a CBDC for different values of the cost of carrying the CBDC. As an example, if the monetary cost of carrying a CBDC relative to cash is around 0.25% of the transaction value, introducing the CBDC will lead to an increase in consumption of 0.04%-0.07% for Canada and 0.12%-0.21% for the US.

In an extension of the model, I assume that the agents' privacy with respect to the size of their transaction should be protected. That is, the planner's power toward CBDC is limited in that the planner cannot see agents' types, whether they want to consume a low- or high-level of consumption good, so the planner can see only the agents' CBDC balances. Interestingly, welfare under the cash-only or co-existence scheme is identical with or without private information, but welfare under the CBDC-only scheme is lower with private information compared with complete information. Altogether, cash is more likely to be used in the optimal scheme and the CBDC-only scheme is less likely to be optimal.

In another extension, I assume that a CBDC is a not a perfect substitute for cash in that the CBDC cannot be used in a fraction of transactions in which cash can be used. In this case, if the cost of carrying the CBDC is low, then co-existence might be optimal. Also, the first best can be achieved as long as the fraction of meetings in which only the CBDC can

[7] In general, a CBDC provides more information to the central bank, such as whether the agent is a buyer or seller or the size of transaction, but I do not use such information in this paper.

[8] Also, in Rogoff's argument, the policy of negative nominal interest rates is needed for short-run stabilization. In contrast, I analyze the steady state of the model, and the interests paid on CBDC balances are aimed at maximizing the long-term welfare of the population, not stabilizing the economy in the short run.

be used and also the discount factor are sufficiently high. This result shows, interestingly, that the CBDC helps to achieve the first best even for the meetings in which only cash can be used (through an OMO).

We see in practice that the governments raise large amounts of revenues from taxes, but central banks do not. One advantage of this model compared with many models in the literature, especially those in the New Monetarist literature, is that the central bank is not granted an unrealistic power with respect to taxation. In the model, the central bank cannot tax agents, not only in the decentralized (anonymous) market, but in the centralized market as well. In other words, my model studies the optimal monetary policy in isolation from fiscal policy, when a central bank cannot count on support from the fiscal authorities, but is free to choose how to use seigniorage revenues. I believe that this is a reasonable description of the current institutional setup in advanced, low-inflation economies.

One may argue that implementing the type of CBDC addressed in this paper is difficult in practice because the interest payments suggested here are traditionally in the realm of fiscal policy, not monetary policy. This argument ignores two facts: First, the central banks in most advanced economies already make interest payments on reserves, but only to some financial institutions that have exclusive access to the central bank facilities. Second, those interest payments are non-linear in that the interest rate paid on reserves is different from the rate charged to borrowers. Central banks have recognized that the payments on reserves in the current system can serve their policy objectives, so why not extend access to all agents if economic efficiency requires that?[9]

The rest of the paper is organized as follows. After briefly discussing the related literature in Section 2, I lay out the model in Section 3. I assume in all sections except Section 8 that cash and a CBDC are perfect substitutes; i.e., both can be used in all transactions. In Section 4 and as a benchmark, I assume cash and a CBDC are both costless to carry. In Section 5, I assume that a CBDC is more costly to carry relative to cash. I show, among other results, that if both cash and the CBDC are used by agents under the optimal policy, cash is used in small-value transactions and the CBDC is used in large-value transactions. In Section 6, I focus on a special case in which there are only two sizes of transactions—large-value and small-value transactions—and characterize conditions under which cash and the CBDC are both used by agents under the optimal policy. I also explore the case in which

[9]Furthermore, I take the implementation of monetary policy more seriously than most papers in monetary economics where the creation of new money is assumed to be done through helicopter drop (lump-sum transfers). In my model, the implementation is done either through an OMO or through direct transfers to CBDC accounts.

agents' privacy should be protected. In Section 7, I calibrate the model to the Canadian and the US data to estimate the welfare gains of introducing a CBDC into these economies. In Section 8, I assume that cash and a CBDC are not perfect substitutes in that in a fraction of meetings, only cash can be used, and in a fraction of meetings, only the CBDC can be used. I show here that co-existence may be welfare enhancing relative to cash-only or CBDC-only schemes. Section 9 concludes.

2 Related Literature

This paper is related to New Monetarist literature, which emphasizes on the micro-foundations of money. The model is built on the framework developed by Lagos and Wright (2005) and Rocheteau and Wright (2005), and has the same structure of a centralized market (CM) and a decentralized market (DM). The CBDC in my paper is similar to the interest-bearing money in Andolfatto (2010). However, he does not study the endogenous choice of means of payment when (the non-interest-bearing) cash and the interest-bearing money are both available to agents; nor does he have idiosyncratic preference shocks that lead to endogenous adoption of different means of payment by agents with different transaction needs.

A closely related paper is Chiu and Wong (2015). They show that electronic money allows the first-best allocation to be implemented under a broader set of parameter values relative to cash. Another related paper is Gomis-Porqueras and Sanches (2013), in which there are two payment systems—fiat money and credit—and there is a cost effectively incurred by buyers to access the credit system. The credit system in their paper has some similarities to the CBDC in my paper. I elaborate in the appendix on the differences between my paper and these two papers.[10]

Dong and Jiang (2010) show that two monies can expand the set of parameters for which the first best is achievable in an environment in which agents have private information about their preference types. Zhu and Hendry (2017) study currency competition between cash and privately issued digital currency. They show that if the private issuer is not welfare maximizing, there will be coordination problems between the central bank and the private issuer. Fernández-Villaverde and Sanches (2016) study a model of competition between several private issuers of fiat currencies. They show that the efficient allocation cannot be

[10] Using credit is not possible in my model, as the central bank cannot see or keep track of agents' actions in the DM. The central bank observes only the agents' CBDC balances at the end of the CM. For other papers in the literature where money and credit are studied in the same model, see Gu et al. (2016) and Chiu et al. (2012) among others.

implemented without intervention, although price stability may be achieved.[11] My model is also related to Rocheteau et al. (2014) in that in both papers, an OMO is used. In my model, an OMO serves as a cross-subsidization device between cash and CBDC users. Finally, on estimating the costs and benefits of issuing CBDC, Barrdear and Kumhof (2016) estimate that CBDC issuance could increase GDP by as much as 3%, mostly through lowering the real interest rates.

3 Model

The model is based on Lagos and Wright (2005) with two means of payment: cash and CBDC. I use index c to refer to cash and index e (for **e**lectronic money) to refer to CBDC. Time is discrete: $t = 0, 1, 2, ...$ Each period consists of two subperiods: DM and CM. In the DM, a decentralized market, and in the CM, a centralized market, is active. There is a continuum of buyers and continuum of sellers, each with a unit mass. Both have discount factor $\beta \in (0, 1)$ from CM to DM. In the the CM, both can consume and produce. In the DM, sellers can only produce and buyers can only consume. In the the CM, one unit of labor supply produces one unit of perishable consumption good. In the DM, a buyer and seller meet randomly with probability σ and split the gains from trade based on proportional bargaining. The buyer's utility function is given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t (w_t u(q_t) + X_t),$$

where buyer's preference shock, w_t, is an i.i.d. draw across time and agents from CDF $F(w)$ and $w \in [w_{min}, w_{max}]$, $u(q)$ is the utility of consuming q units of the DM good, and X_t is the consumption of numeraire in the the CM. Introducing the preference shock is the first departure from the standard Lagos-Wright model. The seller's utility function is given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t (-c(q_t) + X_t).$$

[11] A growing body of literature studies CBDC and its implications for payment systems, monetary policy implementation and financial stability. I cannot do justice to all papers in this literature, but to mention only some examples, Fung and Halaburda (2016) study a framework to assess why a central bank should issue digital currency. Davoodalhosseini and Rivadeneyra (2020) propose a policy framework to evaluate the tradeoffs that policy makers face regarding different types of electronic money, including a CBDC. Kahn et al. (2017) study different schemes of CBDC and discuss how these schemes can meet the central bank's objectives. Finally, Berentsen and Schar (2018) argue in favor of central banks issuing CBDC. In particular, they argue that implementing monetary policy using CBDC is more transparent than the current way of implementing monetary policy.

where $c(q)$ is the cost of producing q units of the DM good. Sellers do not receive a preference shock. All interesting actions come from the buyers' decisions in this paper. I assume that $u'' < 0 < u', u(0) = 0$ and $0 < c'', 0 < c', c(0) = 0$. Clearly, the first-best production level for type w is given by:

$$q_w^* = \arg\max_q \{wu(q) - c(q)\}.$$

Another departure from the standard Lagos-Wright model is that there are two means of payment in this economy—cash and CBDC. Denote by $c_e(z_e) : \mathbb{R}_+ \to \mathbb{R}_+$ the cost of carrying z_e units of real balances (in terms of the the CM good) in the form of CBDC from CM to DM. It is assumed that the buyer incurs this cost. The cost of carrying real balances in the form of cash is assumed to be zero.

The timing of actions and realization of shocks in period t are specified as follows. In the DM, agents are randomly matched and trade according to the proportional bargaining protocol, with $\theta \in [0, 1]$ being the share of the buyer. In the bilateral meeting, w is known both to the buyer and seller, so there is no problem regarding private information. After agents trade in the DM and get separated from the match, the buyers learn their w for the next period. Next, agents trade in the the CM. They work and choose the number of CBDC and cash balances they want to carry to the next DM. At the end of the the CM, new cash and CBDC are transferred to agents as will be described below. Denote by $z_c \in \mathbb{R}_+$ the number of pre-transfer real balances in the form of cash. Similarly, denote by $z_e \in \mathbb{R}_+$ the number of pre-transfer real balances in the form of CBDC. Denote by $t_c \in \mathbb{R}_+$ the helicopter drop of cash in real terms (units of the the CM good) to all buyers. Denote by $t_e(z_e, w) : \mathbb{R}_+ \to \mathbb{R}_+$ the number of CBDC transfers in real terms to type w buyers that have brought z_e from the the CM. It is assumed that the planner has complete information about the buyer's type if the buyer uses CBDC. Yet, the only feasible policy using cash is a helicopter drop. This assumption requires that the preference shock that buyers receive becomes known to the planner when CBDC is used, but the planner cannot identify people when cash is used.[12] Given the policy described above, post-transfer cash and CBDC balances are given by $z_c(w) + t_c$ and $z_e(w) + t_e(z_e(w), w)$. The following notation will be used in the rest of the paper: $x \equiv x_t$ and $x_+ \equiv x_{t+1}$.

[12] The preference shock is known to the planner when buyers use CBDC. This assumption can be motivated by the fact that the number of CBDC balances brought to the bilateral meetings can be used at the end of the DM to verify the agent's preference shock. Even if w is not observable to the planner, the insight that more efficient allocations can be implemented by using CBDC than with cash remains, although the cash is more likely to be part of an optimal scheme. See Section 6.2 for an extension where the planner cannot see agents' types.

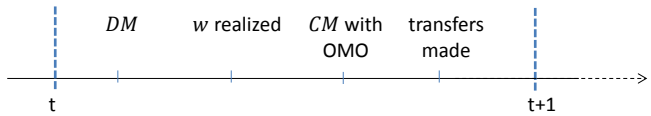

Figure 1: Timing

The growth rates for cash and CBDC supply are denoted by $\gamma_c > 0$ and $\gamma_e > 0$, respectively, so

$$M_{t+1} = \gamma_c M_t, \quad E_{t+1} = \gamma_e E_t,$$

where M_t and E_t denote the cash and CBDC stocks, respectively, at the beginning of the the CM at time t. Each buyer is endowed with the steady state level of cash and CBDC in the DM of $t = 0$.

There is a rationale for both fixed and flexible exchange rates. Under the fixed exchange rate, the inflation rates for cash and CBDC are the same, while they can be different under the flexible exchange rate. On one hand, one dollar issued by the central bank has traditionally had the same value regardless of whether it is in the agent's pocket in the form of cash or with their account in electronic form. On the other hand, there is no reason why this should be the case. As a fixed exchange rate for domestic versus foreign currencies was a dominant paradigm at some point and then partially or completely abandoned, so why not let the exchange rate between cash and CBDC be flexible too, should efficiency require? In this paper, I allow for a flexible exchange rate.[13]

We focus on the cases where total real cash and CBDC balances are constant over time: $\phi_t M_t = \phi_{t+1} M_{t+1}$ and $\psi_t E_t = \psi_{t+1} E_{t+1}$. They imply

$$\phi_t = \gamma_c \phi_{t+1}, \quad \psi_t = \gamma_e \psi_{t+1}. \tag{1}$$

I allow the planner to use an Open Market Operation, OMO, to change the relative supply of cash and CBDC. By an OMO, I mean that the government trades CBDC for cash in the

[13]It is shown in the proofs that CBDC inflation rate is irrelevant as long as it is higher than a threshold. As a result, if the optimal cash inflation is lower than that threshold, then imposing a fixed exchange rate will be a binding restriction for the planner's problem, leading to a less efficient allocation. In a related discussion, Agarwal and Kimball (2015) argue that if cash and CBDC co-exist, allowing for the exchange rate to be different from par makes it possible to implement a negative interest rate policy. My paper and theirs share the feature that if a flexible exchange rate between cash and CBDC is allowed, the outcome is more efficient compared to that with a fixed exchange rate, at least under some parameters. However, my paper is about long-run policies, not short-run stabilization policies.

CM with the price $\frac{\psi}{\phi}$. In that case, the equilibrium conditions can be written as follows:

$$\phi(\bar{M} - M) + \psi(\bar{E} - E) = 0, \tag{2}$$

$$t_c = \phi_+(M_+ - \bar{M}), \tag{3}$$

$$\int t_e(z_e(w), w)dF(w) = \psi_+(E_+ - \bar{E}), \tag{4}$$

$$\int z_c(w)dF(w) = \phi_+\bar{M}, \tag{5}$$

$$\int z_e(w)dF(w) = \psi_+\bar{E}, \tag{6}$$

where $\bar{M}$ and $\bar{E}$ are the cash and CBDC supplies after an OMO and before transfers are made to the agents, and $z_c(w)$ and $z_e(w)$ are real balances of cash and CBDC that a buyer of type w holds.

Equation (2) states that the *net* number of real balances supplied to the the CM in the form of cash and CBDC is equal to zero. It shows that an OMO is a cross-subsidization tool between cash and CBDC users. Without an OMO, $\bar{M} = M$ and $\bar{E} = E$, so t_c will be pinned down by the inflation rate of cash. With an OMO, the amount of cash distributed among agents in the transfer stage can be less than the amount of newly created cash, allowing the planner to use a fraction of newly created cash in an OMO to potentially achieve better allocations. Equations (3) and (4) simply pin down the value of transfers in the form of cash and CBDC, respectively, available to be distributed across agents. For example, t_c is the real value of balances in the CM_{t+1} given to buyers in the transfer stage of period t. Equations (5) and (6) are market-clearing conditions for cash and CBDC.

Lemma 1. *With an OMO, the following constraint should hold:*[14]

$$t_c + \int t_e(z_e(w), w)dF(w) = (\gamma_c - 1)\int z_c(w)dF(w) + (\gamma_e - 1)\int z_e(w)dF(w). \tag{7}$$

3.1 Agents' Problems in the the CM

Buyer's problem in the the CM:

$$W_w^B(z) = \max_{X,Y,z_c,z_e} \left\{ X - Y - c_e(z_e + t_e(z_e, w)) + \beta V_w^B(z_c + t_c, z_e + t_e(z_e, w)) \right\}$$

[14]This constraint is consolidated for both cash and CBDC. Without an OMO, this condition should be replaced by the following two constraints: $t_c = (\gamma_c - 1)\int z_c(w)dF(w)$ and $\int t_e(z_e(w), w)dF(w) = (\gamma_e - 1)\int z_e(w)dF(w)$. In this case, the gains of CBDC would be more limited.

$$\text{s.t. } X + \gamma_c z_c + \gamma_e z_e = Y + z,$$

where z denotes the real balances that the buyer has at the beginning of the the CM. Also, $V_w^B(z_c, z_e)$ is the value function in the DM of the buyer of type w with z_c real balances in cash and z_e real balances in CBDC. Incorporating the constraint into the objective function, one can write:

$$W_w^B(z) = z + \max_{z_c, z_e}\{-\gamma_c z_c - \gamma_e z_e - c_e(z_e + t_e(z_e, w)) + \beta V_w^B(z_c + t_c, z_e + t_e(z_e, w))\}. \quad (8)$$

The sellers' value function in the the CM can be written similarly.

3.2 Agents' Problems in the DM

Buyers receive:

$$V_w^B(z_c, z_e) = \mathbb{E}W_w^B(z_c + z_e)$$
$$+\sigma\Big(wu(q_w(z_c, z_e)) + \mathbb{E}W_w^B(z_c + z_e - d_{c,w}(z_c, z_e) - d_{e,w}(z_c, z_e)) - \mathbb{E}W_w^B(z_c + z_e)\Big)$$
$$= \mathbb{E}W_w^B(z_c + z_e) + \sigma\Big(wu(q_w(z_c, z_e)) - d_{c,w}(z_c, z_e) - d_{e,w}(z_c, z_e))\Big),$$

where $q_w(z_c, z_e)$, $d_{c,w}(z_c, z_e)$, $d_{e,w}(z_c, z_e)$ denote, respectively, the production amount and the real transfer of cash and CBDC balances in the DM meetings in which the buyer has brought z_c real balances in cash and z_e real balances in CBDC. Also, the expectation is taken over realizations of buyer types in the next period. Similarly, sellers receive:

$$V_w^S(z_c, z_e) = W^S(z_c + z_e) + \sigma\Big(- c(q_w(z_c, z_e)) + d_{c,w}(z_c, z_e) + d_{e,w}(z_c, z_e)\Big).$$

Superscript S represents the seller's associated variable. The linearity of W^S and W^B were used to simplify the DM value functions. Sellers do not need to bring balances to the DM because carrying balances is costly and the sellers do not use them until the next the CM. Therefore, we focus only on the buyer's balances, determined from the bargaining protocol.

3.3 Proportional Bargaining in the DM

Terms of trade are determined from the following maximization problem:

$$\max_{q, d_c \in [-z_c^S, z_c], d_e \in [-z_e^S, z_e]} \Delta^B + \Delta^S$$
$$\text{s.t.: } \Delta^B = \theta(\Delta^B + \Delta^S),$$

where Δ^B and Δ^S denote buyer's and seller's surplus, respectively, and are given by:

$$\Delta^B \equiv V_w^B(z_c - d_c, z_e - d_e) - V_w^B(z_c, z_e) + wu(q),$$

$$\Delta^S \equiv V_w^S(z_c^S + d_c, z_e^S + d_e) - V_w^S(z_c^S, z_e^S) - c(q).$$

The solution to the bargaining problem is given by:[15]

$$d_{c,w}(z_c, z_e) + d_{e,w}(z_c, z_e) = \min\Big\{z_c + z_e, D_w(q_w^*)\Big\},$$

$$q_w(z_c, z_e) = D_w^{-1}\Big(d_{c,w}(z_c, z_e) + d_{e,w}(z_c, z_e)\Big),$$

where $D_w(.)$ is defined as follows:

$$D_w(q) \equiv \theta c(q) + (1-\theta)wu(q).$$

Equivalently, the solution is given by:

$$(q_w, d_{c,w} + d_{e,w}) = \begin{cases} (q_w^*, D_w(q_w^*)) & \text{if } z_c + z_e \geq D_w(q_w^*) \\ (D_w^{-1}(z_c + z_e), z_c + z_e) & \text{otherwise} \end{cases}. \tag{9}$$

In words, $D_w(q)$ denotes the number of real balances that a type w buyer needs for buying q units of the DM good. If the buyer brings at least $D_w(q_w^*)$, then the first best is achievable; i.e., the first-best level of production, q_w^*, can be produced. Otherwise, the buyer spends the entire balances, and the terms of trade are given by the second line above. Finally, the value function for buyers and sellers at the beginning of the DM can be written as follows:

$$V_w^B(z_c, z_e) = \mathbb{E}W_w^B(z_c + z_e) + \sigma\theta\Big(wu(q_w(z_c, z_e)) - c(q_w(z_c, z_e))\Big),$$

$$V_w^S(z_c, z_e) = W^S(z_c + z_e) + \sigma(1-\theta)\Big(wu(q_w(z_c, z_e)) - c(q_w(z_c, z_e))\Big).$$

3.4 The CM and DM Value Functions Together

The buyer's problem turns into:

$$W_w^B(z) = z + \beta\mathbb{E}W_w^B(0)$$

$$+\max_{z_c, z_e}\Big\{-\gamma_c z_c - \gamma_e z_e - c_e(z_e + t_e(z_e, w)) + \beta(z_c + t_c) + \beta(z_e + t_e(z_e, w)) + \beta\sigma\theta(wu(q) - c(q))\Big\}, \tag{10}$$

[15] This problem is basically the same as follows: $\max_{x,d\in[-z',z]}[u(x) - x]$ subject to $u(x) - d = \theta(u(x) - x)$.

where q is implicitly given by $D_w(q) = \min\{D_w(q_w^*), z_c + t_c + z_e + t_e(z_e, w)\}$.

It is standard to show that sellers do not bring any balances to the DM. In the DM, if sellers get matched, they work to produce the DM good and sell it to the buyer, and then bring their balances to the the CM and use them to purchase the the CM good and consume it. Buyers work to acquire money (cash or CBDC) in the the CM, and receive transfers from the planner. Then, they enter the DM with the entire money stock, and exchange it all for the DM good if matched.

3.5 Equilibrium Definition

The equilibrium definition can now be written as follows.

Definition 1 (Stationary Equilibrium)**.** *Stationary equilibrium is a price system $\{\phi_t\}$, $\{\psi_t\}$, an allocation $\{(q(w), z_c(w), z_e(w))\}_w$ and a policy $\{\gamma_c, \gamma_e, t_c, \{t_e(z_e, w)\}_{z_e,w}\}$ such that the following conditions hold:*

(i) Buyer's maximization in the the CM: $z_c = z_c(w)$ and $z_e = z_e(w)$ solve (8) for a given $z_{c,0}$ and $z_{e,0}$ (initial values of z_c and z_e), $\{\psi_t\}_{t=0}^{\infty}$ and $\{\phi_t\}_{t=0}^{\infty}$.

(ii) Market clearing for cash and CBDC, planner's budget constraint and OMO's equation: (7) should hold.

(iii) Proportional bargaining: $q_w = q(w)$ solves (9).

(iv) Growth equation (1) for cash and CBDC.

3.6 Planner's Problem

The planner's problem is to maximize welfare, calculated at the beginning of the the CM, by choosing a policy:

Problem 1 (Planner's Problem)**.**

$$\max_{\{\gamma_c,\gamma_e,t_c,\{t_e(z_e,w)\}_{z_e,w}\}} \int \Big[\beta\sigma\big(wu(q(w)) - c(q(w))\big) - c_e(z_e(w))\Big] dF(w)$$

subject to: $\{(q(w), z_c(w), z_e(w))\}_w$ form an equilibrium together with some prices $\{\phi_t\}$, $\{\psi_t\}$ and the policy.[16]

[16] One may want to consider the planner's problem without β, only to focus on the surplus created in the economy without worrying about time preferences from the CM to DM. Even without β, all results will continue to hold; one just needs to replace the cost of carrying CBDC with $c_e(.)/\beta$.

Throughout the paper, I assume that the cost of carrying CBDC is flat.

Assumption 1 (Cost Function)**.** *CBDC costs $K \geq 0$ in terms of the the CM good and is to be incurred in the the CM. That is, $c_e(z) = K\mathbb{I}\{z > 0\}$.*

As will be shown later, if both cash and CBDC are costless, cash is redundant, because CBDC is a more powerful instrument for the planner to implement monetary policy. For the planner to have a non-trivial problem regarding which means of payment should be available to agents, CBDC needs to be disadvantageous to cash in some ways. This disadvantage is motivated by the fact that agents in the economy may value anonymity while doing transactions, and they may lose it if they use CBDC. Also, electronic means of payment including CBDC usually require some devices to process the payments, while cash does not, so this cost can summarize the costs of using such devices. This disadvantage is modeled for simplicity as a flat cost $K \geq 0$ for carrying CBDC. The flat cost of carrying CBDC is consistent with the digital format of CBDC in that the dis-utility of losing anonymity for the agent may be independent of the number of balances that the agent holds.

3.6.1 Simplified Planner's Problem

The main constraint for the planner's problem, Equation (10), can be written as:

$$(q(w), z_c(w), z_e(w)) \in \arg \max_{q \in [0, q_w^*], z_c, z_e} \begin{bmatrix} -(\gamma_c - \beta)(z_c + t_c) - (\gamma_e - \beta)(z_e + t_e(z_e, w)) \\ -c_e(z_e + t_e(z_e, w)) + \gamma_c t_c + \gamma_e t_e(z_e, w) \\ +\beta\sigma\theta(wu(q) - c(q)) \end{bmatrix}, \tag{11}$$

$$\text{s.t. } D_w(q) = \min\{D_w(q_w^*), z_c + t_c + z_e + t_e(z_e, w)\}. \tag{12}$$

Given the assumption on the cost function, the planner's problem can be simplified as follows:

Problem 2.

$$\max_{\{\gamma_c, \gamma_e, t_c, \{t_e(z_e, w)\}\}} \int \left[\beta\sigma\Big(wu(q(w)) - c(q(w))\Big) - K\mathbb{I}(z_e(w) > 0)\right] dF(w)$$

$$(q(w), z_c(w), z_e(w)) \textit{ taken from } (11) \textit{ and } (12),$$

$$\textit{and } t_c + \int (t_e(z_e(w), w) - (\gamma_c - 1)z_c(w) - (\gamma_e - 1)z_e(w)) dF(w) = 0.$$

Proposition 1. *In the solution to the planner's problem, one can assume without loss of generality that $t_e(z, w)$ is a step function in z with a sufficiently large γ_e. That is,*

$$t_e(z, w) = \begin{cases} t_{0,w} & z \geq z_{0,w} \\ 0 & z < z_{0,w} \end{cases}$$

for some $t_{0,w} \in \mathbb{R}_+, z_{0,w} \in \mathbb{R}_+$.

This proposition states that we can restrict our attention to the CBDC transfer schemes that are step functions. That is, if agents of type w brings at least $z_e(w)$, then they receives some transfers, but bringing any lower real balances in CBDC does not yield them any transfers.[17]

In the next section, I study the case in which cash and CBDC are costless as a benchmark ($K = 0$). Next, I study the case in which CBDC is more costly than cash ($K > 0$).

4 Special Case: Costless CBDC

I show that if $K = 0$, cash is redundant. I also study conditions under which first best is achievable with CBDC. It is impossible to achieve the first best with only cash, because it is not possible to tax cash holdings nor to make transfers to agents based on their cash holdings.

Proposition 2 (Redundancy of cash when $K = 0$)**.** *If both cash and CBDC are costless, then cash is redundant. That is, any allocation that is achieved by using cash and CBDC can be achieved by using only CBDC.*

The idea is that if both cash and CBDC are costless, CBDC has a clear advantage for the planner, as the planner can provide incentive for buyers to bring enough balances to the DM by checking their CBDC balances. The planner can then punish agents who do not bring enough balances from the the CM by making zero transfers to them. The cash growth rate is set sufficiently high that the gains from using cash and consequently the demand for cash become zero.

[17] What mechanisms can guarantee that agents do not want to pool resources? The planner can easily put a cap on money holdings eligible for transfers. If the agent's money holding exceeds a certain threshold, the agent would not receive any transfers.

4.1 Homogeneous Buyers

The planner sees w and can make transfers to buyers who use CBDC contingent on their types. Cash can be distributed across agents only evenly (helicopter drop). In this section, there is only one type so only one means of payment is generally used.

Proposition 3. *Suppose both cash and CBDC are costless; i.e., $K = 0$. Suppose also the distribution of types is degenerate at w; i.e., there is only one type. The first best is achievable if and only if:*

$$\beta\sigma\theta(wu(q_w^*) - c(q_w^*)) \geq (1-\beta)D_w(q_w^*).$$

The left-hand side (LHS) of the condition is the buyer's gains from bringing the number of balances that the planner asks for. The right-hand side (RHS) is the real cost of holding balances. This is the inevitable cost of holding real balances with CBDC: Carrying CBDC for buying q units of the DM good imposes $(\gamma_e - \beta)D_w(q_w^*)$ cost of real balances on the buyer, but the newly created CBDC will be distributed across buyers such that they receive $(\gamma_e - 1)D_w(q_w^*)$ real balances if they have brought enough balances. Therefore, they have to incur the cost $(1-\beta)D_w(q_w^*)$. Noticeably, the inflation rate of CBDC does not affect the incentives.

The condition required by this proposition is equivalent to:

$$\theta \geq \overline{\theta}(w) \equiv \frac{1-\beta}{(1-\beta(1-\sigma))(1-\frac{c(q_w^*)}{wu(q_w^*)})}.$$

For $\overline{\theta}(w) \leq 1$, we must have:

$$\beta \geq \left(1 + \sigma\frac{wu(q_w^*) - c(q_w^*)}{c(q_w^*)}\right)^{-1}.$$

To achieve the first best, the proposition requires θ and also β to be sufficiently high. Even in the case of $\theta = 1$, in which the buyer takes the entire surplus, the buyer still needs to work in the CM_t to earn $c(q)$ real balances in CBDC. The benefits of CBDC will be realized in the DM_{t+1} with probability σ, in the DM_{t+2} with probability $(1-\sigma)\sigma$, and so on. For the benefits to dominate the costs, one needs: $wu(q)(\beta\sigma + \beta^2(1-\sigma)\sigma + \beta^3(1-\sigma)^2\sigma + ...) \geq c(q)$, which is equivalent to the above condition.[18]

The following remark implies that for a given set of parameters, a threshold for w exists below which the first best cannot be achieved and above which the first best can be achieved.

Remark 1. *$\bar{\theta}(w)$ is decreasing in w if $\frac{c'(q)u(q)}{c(q)u'(q)}$ is increasing in q.*

[18] Chiu and Wong (2015) provide this intuitive explanation for a related discussion.

The required condition is satisfied, for example, when $c(q)$ is linear and $u(q) = \frac{(q+b)^{1-\eta}-b^{1-\eta}}{1-\eta}$ where $\eta \in (0,1)$ and $b > 0$.

4.2 Heterogeneous Buyers

Now suppose that the distribution of types is not degenerate. If the population is composed of only individual types, the first best may be achieved for some types and not for others. The following proposition provides sufficient conditions to achieve the first best when we have a non-degenerate distribution of those types together.

Proposition 4. *Suppose both cash and CBDC are costless; i.e., $K = 0$. With heterogeneous types, the first best is achievable if and only if:*

$$\beta\sigma\theta \int (wu(q_w^*) - c(q_w^*))dF(w) \geq (1-\beta) \int D_w(q_w^*)dF(w).$$

Compared with the case with homogeneous buyers, cross-subsidization is possible here. This can be seen clearly from comparing the conditions in Propositions 3 and 4. The idea is that if the condition in Proposition 3 is slack for some types, say w_2, and does not hold for other types, say w_1, the planner can charge type w_2 buyers to subsidize type w_1 buyers so that the latter buyers bring enough balances to the DM. It is not possible to cross-subsidize with cash, because it is not possible to see types or the amount of balances.

I emphasize that types are assumed to be observable in the CBDC system, so cross-subsidization is possible and welfare enhancing. Without observability of types, the higher types may want to pretend to be of a lower type. I analyze that case for a two-type example in Section 6.2 and show that the gains from CBDC will be reduced.

5 Costly CBDC

In this section, consider the case in which cash is still costless but CBDC requires flat cost $K > 0$ in real balances to carry from the the CM to the DM. CBDC is costly, so cash may not be redundant anymore and the planner may want some types to use cash. An important task is to characterize the types who use cash and the types who use CBDC. Cash inflation is costly for those agents who carry cash. However, it may still be optimal for them to bring cash because there is a direct cost associated with carrying CBDC. Thus, the interesting tradeoff here is whether the planner should increase cash inflation so as to encourage more buyers to use CBDC and achieve better allocations, or decrease cash inflation to have less distorted allocation for cash users and to save on CBDC carrying costs. Since the CBDC

cost is independent of the amount of CBDC that buyers carry, if the planner wants a buyer to carry some CBDC, the planner wants that buyer to carry his or her entire balances in the form of CBDC.

I introduce the following notation, which will prove useful in the rest of the paper:

$$\begin{aligned}
f(w,q) &\equiv wu(q) - c(q), \\
s(w,q) &\equiv -(1-\beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q)), \\
O(w,\gamma) &\equiv \max_q\{-(\gamma-\beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\}, \\
\bar{q}(w,\gamma) &\equiv \arg\max_q\{-(\gamma-\beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\}, \\
e(w,\gamma) &\equiv \begin{cases} 1 & \text{type } w \text{ uses CBDC} \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

Function $f(w,q)$ is the surplus created in a match of a buyer of type w who consumes q units of the DM good. Function $s(w,q)$ is the present value of the payoff that a buyer of type w receives in the the CM from working for (and holding) $D_w(q)$ units of real balances that can be used to buy q units of the DM good when the inflation rate is zero, assuming that only cash is available. Function $O(w,\gamma)$ is the maximum present value of the payoff that a buyer of type w can receive when the inflation rate is $\gamma - 1$ and $\bar{q}(w,\gamma)$ is the associated consumption of the DM good, again assuming that only cash is available. Finally, $e(w,\gamma)$ is simply an indicator function for a buyer of type w when the cash inflation rate is $\gamma - 1$ (and CBDC inflation rate is sufficiently high). It takes the value of 1 if the buyer uses CBDC and takes the value of 0 otherwise.

5.1 Homogeneous Buyers

Similar to the last section, I begin by analyzing the case for homogeneous buyers. Since there is no heterogeneity and the cost of carrying CBDC is flat, either all buyers use cash or all use CBDC. As a result, it suffices to calculate the highest possible welfare under cash and under CBDC separately and then compare them. Define $\tilde{e}(w)$ as follows. If only CBDC is used, then $\tilde{e}(w) = 1$, and if only cash is used, then $\tilde{e}(w) = 0$.

First, suppose buyers use CBDC. From the constraints in the planner's problem, we have $t_e = (\gamma_e - 1)z_e$ and $t_e + z_e = D_w(\tilde{q})$ where $\tilde{q}$ is the DM consumption under CBDC. Therefore, $t_e = (\gamma_e - 1)D_w(\tilde{q})/\gamma_e$. Note that t_c is set to zero because distributing cash would only distort the allocation. The planner's problem can be written as follows:

$$\max_{\tilde{q}} \left\{\beta\sigma f(w,\tilde{q}) - K\right\}$$

s.t. $-(1-\beta)D_w(\tilde{q}) + \beta\sigma\theta(wu(\tilde{q}) - c(\tilde{q})) - K \geq \max_{\tilde{q}}\{-(\gamma_c - \beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\}$.

The cash inflation rate, $\gamma_c - 1$, is chosen to be sufficiently high that the RHS of the constraint becomes zero. Also, $\gamma_e - 1$ must be sufficiently large so that CBDC transfers become positive. Denote by $\tilde{q}(w)$ the solution to this problem. It is easy to see that if $K \leq s(w, q_w^*)$, then $\tilde{q}(w) = q_w^*$. If $K > s(w, q_w^*)$, then $\tilde{q}(w)$ is implicitly given by $K = -(1-\beta)D_w(\tilde{q}) + \beta\sigma\theta(wu(\tilde{q}) - c(\tilde{q}))$. In this case, obviously, $\tilde{q}(w) < q_w^*$.

Second, suppose buyers use cash, then it is optimal to set the cash inflation rate to the lowest possible level; i.e., $\gamma_c = 1$. The value of the objective function then equals $\beta\sigma f(w, \bar{q}(w, 1))$.

Therefore, it is optimal to use CBDC if and only if $\beta\sigma f(w, \bar{q}(w, 1)) < \beta\sigma f(w, \tilde{q}(w)) - K$. The following proposition summarizes this discussion.

Proposition 5. *The production level, $\tilde{q}(w)$, and the optimal choice of means of payment for the case in which all buyers are of type w, $\tilde{e}(w)$, can be summarized as follows:*

$$\text{if } K_1(w) \leq K_2(w), \text{ then: } \begin{cases} \tilde{q}(w) = q_w^*, \tilde{e}(w) = 1 & K \leq K^*(w) \equiv K_1(w) \\ \tilde{q}(w) = \bar{q}(w, 1), \tilde{e}(w) = 0 & K > K^*(w) \end{cases},$$

$$\text{if } K_1(w) > K_2(w), \text{ then: } \begin{cases} \tilde{q}(w) = q_w^*, \tilde{e}(w) = 1 & K \leq K_2(w) \\ \tilde{q}(w) < q_w^*, \tilde{e}(w) = 1 & K_2(w) < K \leq K^*(w) \\ \tilde{q}(w) = \bar{q}(w, 1), \tilde{e}(w) = 0 & K > K^*(w) \end{cases},$$

where

$$\begin{aligned} K_1(w) &\equiv \beta\sigma f(w, q^*(w)) - \beta\sigma f(w, \bar{q}(w, 1)), \\ K_2(w) &\equiv s(w, q_w^*), \end{aligned}$$

and $K^*(w)$ denotes the cost threshold at which the planner is indifferent between the schemes in which only cash is used by everyone or only CBDC is used by everyone.

According to this proposition, CBDC is used when K is small and cash is used when K is large. Interestingly, when $K_1(w) > K_2(w)$, although CBDC is optimally used for $K \in (K_2(w), K^*(w))$, but the first best cannot be achieved. See Figure 2 for illustration.

When $\tilde{e}(w) = 0$, the cash inflation rate is zero—i.e., $\gamma_c = 1$—and transfers are given by $t_e(z, w) = 0$ and $t_c = 0$. When $\tilde{e}(w) = 1$, the transfers are given by:

$$t_e(z, w) = \begin{cases} \frac{(\gamma_e - 1)D_w(\tilde{q}(w))}{\gamma_e} & z \geq \frac{D_w(\tilde{q}(w))}{\gamma_e} \\ 0 & z < \frac{D_w(\tilde{q}(w))}{\gamma_e} \end{cases}.$$

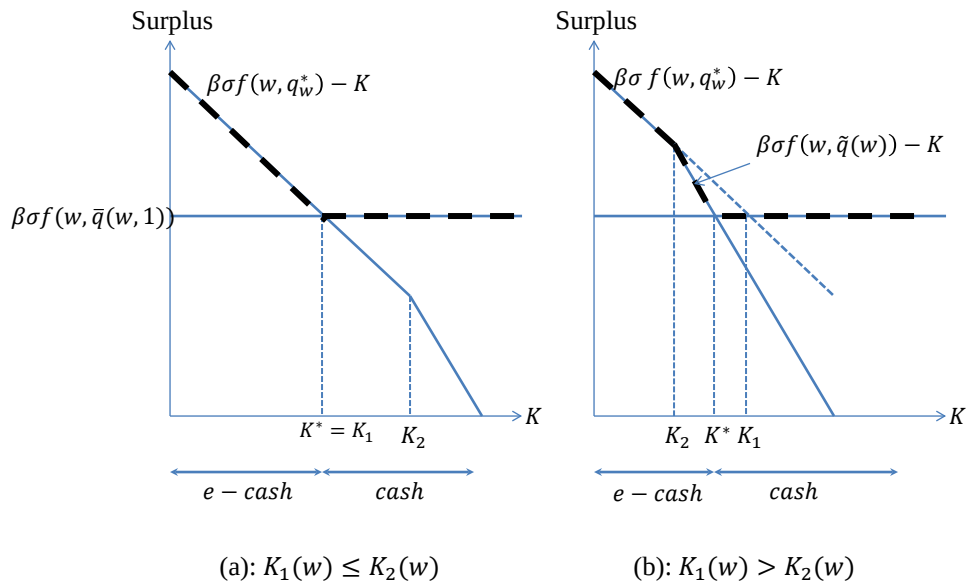

Figure 2: Usage of cash versus CBDC with fixed cost of carrying CBDC

Proposition 6. *There exists* $(\hat{\beta}, \hat{\theta}) \in (0,1)^2$ *such that if* $\beta > \hat{\beta}$ *and* $\theta > \hat{\theta}$*, then* $K_2(w) \geq K_1(w)$*.*

This proposition states that $K_1(w) \leq K_2(w)$ for sufficiently large values of β and θ. In this case, if CBDC is used, then the first best will be achieved.

5.2 Heterogeneous Buyers

We can now compare between homogenous and heterogeneous cases. As in the costless case, cross-subsidization is possible with multiple types, so if the incentive constraint is binding for some types and not for others, cross-subsidization can help to achieve better allocations. The incentive constraint states that CBDC users should gain a weakly higher payoff from using CBDC compared with cash.[19]

With heterogeneity, the planner is more restricted. For those types with $K^*(w)$ greater than K, say type w_2, use of CBDC is optimal when the population is homogeneously composed of w_2. However, if there is a sufficiently high measure of agents with $K^*(w)$ less than

[19] We do not need to consider the constraint that cash users should gain a higher payoff from using cash relative to CBDC, because if they switch to using CBDC, their type would be immediately revealed and they would receive the net payoff of zero.

K (who would like to use cash), say type w_1, then setting a high cash inflation rate amounts to a significant loss in social welfare. As a result, cash inflation cannot be too high. Therefore, type w_2 may want to switch to cash (inefficiently compared with the homogenous case) because the punishment for using cash cannot be severe enough to induce this type to use CBDC.

5.2.1 Cash Is Used in Small-Value Transactions

In the following proposition, we establish that the low-type buyers use cash and high-type buyers use CBDC. That is, cash is used for small-value transactions and CBDC is used for large-value transactions under the optimal policy. Define:

$$Q(w) \equiv \arg\max_q \{wu(q) - c(q)\}.$$

Proposition 7. *Assume* $\frac{wQ''(w)}{Q'(w)} \geq -1$. *Then, there exists a threshold* $w^t > 0$ *such that agents with* $w < w^t$ *use cash and agents with* $w \geq w^t$ *use CBDC under the optimal policy.*

This result is not trivial. Cash inflation is not ∞, so some agents can receive a strictly positive payoff by using cash. Since the size of the surplus is higher for higher types, they receive a higher payoff for a given cash inflation rate. If their payoff from holding cash increases very fast with their type, it may not be worth it for the planner to have these types use CBDC.[20] It is shown that if $\frac{wQ''(w)}{Q'(w)} \geq -1$, this does not happen. This condition states that the coefficient of relative risk aversion of Q should be less than 1. This condition is satisfied for the production and cost functions, u and c, usually used in economics. As an example, let $u(q) = q^{1-1/c_0}$ with $c_0 > 1$ and $c(q) = c_1 q$. Hence, $Q(w) = (\frac{1-1/c_0}{c_1})^{c_0} w^{c_0}$, so $\frac{wQ''(w)}{Q'(w)} = c_0(c_0 - 1)/c_0 = c_0 - 1 > 0 > -1$.[21]

Here, I briefly provide some intuition for this result. It is shown in the proof that the amount of consumption for the types using CBDC is given by the following:

$$\tilde{q}(w) = \arg\max_q \{\beta\sigma f(w,q) + \tilde{\lambda} s(w,q)\},$$

[20] This finding is consistent with facts from a survey in Canada reported by Fung et al. (2015) that cash is used mainly for small-value transactions. They add that the share of cash usage relative to the usage of other means of payment has decreased. Interestingly, they report that the respondents to the survey attribute their cash usage mostly to its lower cost relative to other means of payment. Other factors, such as security concerns, acceptance by the merchants, and ease of use, come after the cost.

[21] More generally, consider a constant relative risk averse $Q(w)$ in which $-\frac{wQ''(w)}{Q'(w)} = 1 - c_0$ where $c_0 \geq 0$. This implies that $Q(w) = k_1 w^{c_0} + k_0$. Therefore, if c and u are such that $\frac{c'(q)}{u'(q)} = (\frac{q-k_0}{k_1})^{1/c_0}$ for some k_0, k_1 and c_0, then the required condition is satisfied.

for some positive $\tilde{\lambda}$. This characterization is useful, because it tells us that given a cash inflation rate, all we need to know for full characterization is $\tilde{\lambda}$, which depends on how tight the constraints in Problem 2 are. The tighter the constraints, the higher the variable $\tilde{\lambda}$. This implies that $\tilde{q}(w) \in [\bar{q}(w,1), q_w^*]$. That is, when agents use CBDC, they can consume more than the consumption level if they were to use cash and less than the first best level. Now, we can explain the role of the condition on Q in Proposition 7. This condition ensures that $\tilde{q}(w)$ increases more than $\bar{q}(w,1)$ with w. As a result, we can show that the gains from introducing CBDC is increasing in the type w, while the cost of carrying CBDC is fixed; therefore, there exists a unique threshold for types. The buyers with a lower type use cash and the buyers with a higher type use CBDC.

6 A Two-Type Example

I focus in this section on a two-type example. Studying this case, as opposed to finitely many types or continuum of types, is more tractable and captures the main tradeoffs. Suppose there are two types w_1 and w_2 with $K^*(w_1) < K < K^*(w_2)$. If the population is homogeneous, type w_1 uses cash and type w_2 uses CBDC under the optimal policy. Denote the measure of type w_2 buyers by π and measure of type w_1 buyers by $1-\pi$. When all buyers are of type w_2, the planner could increase cash inflation so that no buyer uses cash. In contrast, when all the buyers are of type w_1, the planner has to use cash for w_1 and set the cash inflation rate to the lowest possible level, $\gamma_c = 1$. What should the planner do when the optimal means of payment for some of them is different from others?

Denote by E the optimal welfare level if both types use CBDC, denote by C the optimal welfare level if both types use cash, and finally denote by B the welfare level if only w_1 uses cash:

$$E = (1-\pi)\beta\sigma f(w_1, q_1^*) + \pi\beta\sigma f(w_2, q_2^*) - K,$$

$$C = (1-\pi)\beta\sigma f(w_1, \bar{q}_1(1)) + \pi\beta\sigma f(w_2, \bar{q}_2(1)),$$

$$B = (1-\pi)\beta\sigma f(w_1, \bar{q}_1(\gamma_c)) + \pi\beta\sigma f(w_2, q_2) - \pi K,$$

where $\bar{q}_i(\gamma)$ is a short form for $\bar{q}(w_i, \gamma)$. We know from the results in the previous section that it is not optimal for type w_1 to use CBDC and for type w_2 to use cash. Therefore, the planner's problem can be written as $\max\{E, C, \overline{B}\}$ where $\overline{B}$ denotes the optimal welfare level if only type w_1 uses cash, and it is obtained from:

$$\overline{B} \equiv \max_{t_c, t_{e2}, z_{c1}, z_{e2}, \gamma_c, \gamma_e, q_2} B$$

$$\text{s.t. } t_c + \pi(t_{e2} - (\gamma_e - 1)z_{e2}) = (1-\pi)(\gamma_c - 1)z_{c1} \text{ (equivalent to (7))},$$

$$D_{w_2}(q_2) = \min\{D_{w_2}(q^*_{w2}), t_c + t_{e2} + z_{e2}\} \text{ (w_2's payment when using CBDC)},$$

$$D_{w_1}(q_1) = \min\{D_{w_1}(q^*_{w1}), t_c + z_{c1}\} \text{ (w_1's payment when using cash)},$$

$$O(w_2, \gamma_c) \leq -(\gamma_e - \beta)(t_{e2} + z_{e2}) - (\gamma_c - \beta)t_c + \beta\sigma\theta(w_2 u(q_2) - c(q_2)) - K + \gamma_e t_{e2} \text{ (incentive constraint)}.$$

Agents do not want to bring more balances than the number of balances needed to buy the first-best level of production, so the incentive constraint for the maximization problem can then be simplified to:

$$O(w_2, \gamma_c) \leq -(1-\beta)D_{w_2}(q_2) + \beta\sigma\theta f(w_2, q_2) - K + \frac{1-\pi}{\pi}(\gamma_c - 1)D_{w_1}(q_1) - \frac{\gamma_c t_c}{\pi}. \quad (13)$$

See the appendix for the derivation. Interestingly, notice that the term, $\frac{1-\pi}{\pi}(\gamma_c - 1)D_{w_1}(q_1)$, reflects the role of an OMO. A positive cash inflation rate relaxes the incentive constraint, because the real value of newly created cash can be transferred to CBDC users through an OMO. Also, a positive t_c makes the constraint only tighter. Since t_c does not appear in the objective function, it is optimal to set it to the lowest possible value; i.e., $t_c = 0$. Now, we obtain the following result.

Proposition 8 (Optimality of positive inflation)**.** *Suppose $K > 0$. If co-existence is optimal, then the cash inflation rate must be strictly positive; i.e., $\gamma_c > 1$.*

This result states that when co-existence is optimal, the cash inflation must be strictly positive, although running a negative cash inflation rate through an OMO is feasible. If cash inflation rate is negative, then cash should be withdrawn from the the CM, requiring CBDC to be injected into the the CM using an OMO. These CBDC balances should be financed from CBDC users. Moreover, as shown earlier, there is an opportunity cost of using CBDC, as the transfers made to buyers can be used to purchase the DM good only in the next period. (This is as if the effective inflation for CBDC cannot be less than 1.) Finally, CBDC users should incur cost K. Altogether, if $\gamma_c \leq 1$, then CBDC is a strictly dominated choice of payment for buyers, so co-existence is not possible under a negative cash inflation rate.

6.1 Sufficient Conditions for Non-Optimality of Co-Existence

I define three schemes. Under the "cash-only" scheme, both types use cash, under the "CBDC-only" scheme, both types use CBDC, and under the "co-existence" scheme, type w_1 uses cash and type w_2 uses CBDC. Note that under the CBDC-only scheme, the first-best level of production can be achieved (assuming that β and θ are sufficiently high). The

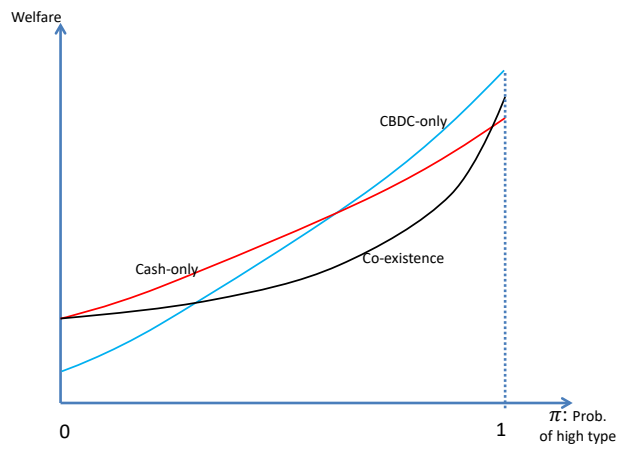

Figure 3: Co-existence is not optimal under conditions specified in Proposition 9.

following proposition provides a sufficient condition for non-optimality of the co-existence scheme, or equivalently, a necessary condition for optimality of co-existence.

Assumption 2. *Assume $\gamma_0 < \gamma_1$, where γ_0 is the smallest $\gamma > 0$ such that*

$$\max_q \{-(\gamma - \beta)D_{w_1}(q) + \beta\sigma\theta f(w_1, q)\} = 0,$$

and γ_1 is implicitly defined by

$$\max_q \{-(\gamma_1 - \beta)D_{w_2}(q) + \beta\sigma\theta f(w_2, q)\} \equiv -(1-\beta)D_{w_2}(q_2^*) + \beta\sigma\theta f(w_2, q_2^*) - K.$$

Proposition 9. *Under Assumption 2, the co-existence scheme is not optimal if π is sufficiently close to 1.*

The schematic diagram for this result can be found in Figure 3. A similar result can be obtained if π is close to zero. The intuition for the proof is as follows. If co-existence is optimal for high values of π, type 1 should receive a positive payoff, so the cash inflation rate, γ, should be lower than γ_0. Using $\gamma < \gamma_0$ and the assumption that $\gamma_1 < \gamma_0$, I show that if type 2 receives the first-best level of consumption, then the incentive constraint will be violated. Therefore, welfare under the co-existence scheme for high values of π is less than that under the CBDC-only scheme where type 2 receives the first-best level of consumption.

Analytical characterization of whether co-existence is optimal for all values of π is not easy. In several numerical examples, many of which are not reported here to save space, the co-existence scheme is shown to lead to a lower welfare level relative to the schemes in which only cash or only CBDC is available to agents. Theoretically, the constraint that

CBDC users should not have incentives to use cash imposes a restriction on the welfare level that the planner can achieve. In the next section, I calibrate the model to the US and Canadian economies and characterize numerically the region of parameters under which the co-existence scheme is optimal.

6.2 Privacy; What If the Planner Cannot See the Agents' Types?

Now assume that agents' privacy should be protected. Particularly, the planner does not see, or is not allowed to use, the agents' types, so the only piece of information available to the planner is CBDC balances. Note that the private information problem is between the buyer and the planner and not between the buyer and the seller, so the bargaining is conducted as in the benchmark model. I re-evaluate the planner's problem under different schemes with newly added constraints stemming from the agents' incentive compatibility (IC) constraints.

Under the cash-only scheme, there is no IC constraint and the welfare level remains the same as in the complete information case, because, by assumption, the only policy toward cash is helicopter drop. Under the co-existence scheme, I show that the IC constraint is not binding, so welfare does not change compared with complete information in this case, either. Under the CBDC-only scheme, however, I show that welfare is reduced compared with the complete information case for certain parameters, implying that the CBDC-only scheme is less likely to be optimal under private information. In other words, when agents have privacy concerns, removal of cash is less likely to be an optimal policy.

In this section, I maintain the assumption that agents can bring from the CM only one means of payment. As noted above, with private information, the IC constraint should be added to the planner's problem. Under the **CBDC-only scheme**, type i should not have incentive to report $j \neq i$. That is, $\Pi_{ii} \geq \Pi_{ij}\ \forall i, j \neq i$, where

$$\Pi_{ij} \equiv -(\gamma_e - \beta)\,(t_j + z_j) + \beta\sigma\theta f_i(q_{ij}) - K + \gamma_e t_j.$$

Notationally, $f_i(q)$ and $D_i(q)$ are short forms for $f(w_i, q)$ and $D_{w_i}(q)$, q_{ij} is given by

$$D_i(q_{ij}) = min\{D_i(q_i^*), t_j + z_j\},$$

and q_{ii} is identical to q_i. The IC constraints can then be simplified to

$$IC_{ij} \text{ (if type } i \text{ reports } j \neq i) : \beta\sigma\theta\left[f_i(q_i) - f_i(q_{ij})\right] \geq (\gamma_e - \beta)(z_i - z_j) - \beta(t_i - t_j) \text{ for both } i.$$

This constraint states that if type i buyers report type j, then they have to bring z_j balances so that the planner accepts their report, in which case they receive t_j balances and can consume q_{ij} units of the DM good.

Similarly, the IC constraint with private information under the **co-existence scheme** can be written as

$$\text{IC}_{12}: \; O(w_1, \gamma_c) \geq -(\gamma_e - \beta)(t_{e2} + z_{e2}) - (\gamma_c - \beta)t_c + \beta\sigma\theta f_1(q_{12}) - K + \gamma_e t_{e2}.$$

Note that under the co-existence scheme, the IC_{21} constraint has been already taken into account in the planner's problem, i.e., (13), so that is not repeated here. I first present the result and then the proof outline. Since the proofs in this section are long, I relegate the details to the Appendix.

Proposition 10 (Optimal scheme with privacy concerns)**.** *In the two-type example, suppose the planner does not see the agents' types.*

(i) Welfare under the ***cash-only scheme*** *with private information is equal to welfare under the cash-only scheme with complete information.*

(ii) For θ sufficiently close to 1, welfare under the ***co-existence scheme*** *with private information is equal to welfare under the co-existence scheme with complete information.*

(iii) Given the following inequalities, welfare under the ***CBDC-only scheme*** *with private information is strictly lower than welfare under the CBDC-only scheme with complete information:*

$$\begin{aligned} \min\left\{ \begin{bmatrix} (1-\pi)f_2(D_2^{-1}(D_1(q_1^*))) \\ +\pi f_2(q_2^*) \end{bmatrix}, f_1(q_1^*) \right\} &< \frac{1-\beta}{\beta\sigma\theta}\left[(1-\pi)D_1(q_1^*) + \pi D_1(q_2^*)\right] + \frac{K}{\beta\sigma\theta} \\ &\leq (1-\pi)f_1(q_1^*) + \pi f_2(q_2^*). \end{aligned} \tag{14}$$

Part (i) of the result is clear. Part (ii) states that under the co-existence scheme, the incentive compatibility for type 1 agents to report type 2, IC_{12}, is not binding. I show it for $\theta = 1$ first, and then use a continuity argument to show that IC_{12} continues to be slack for θ sufficiently close to 1. The reason that I focus on $\theta = 1$ is technical. The consumption of type 1 if this type claims to be type 2 is generally given by $min\{q_1^*, D_1^{-1}(D_2(q_2))\}$, which is difficult to work with, but it is simplified to $min\{q_1^*, q_2\}$ when $\theta = 1$.

Part (iii) states that under under the CBDC-only scheme, the IC_{12} constraint is binding and the welfare is reduced with private information compared with complete information. The first best is achieved with complete information, thanks to the second inequality in (14), but not with private information, due to the first inequality in (14). It is shown in the proof that any allocation that supports the first best is not incentive compatible, i.e., type 1 agents have incentive to bring more resources to claim to be type 2 and receive more transfers under the first best allocation. As a special case, when θ goes to 1, then $D_1(q) = D_2(q) = c(q)$, therefore, the LHS expression in (14) is equal

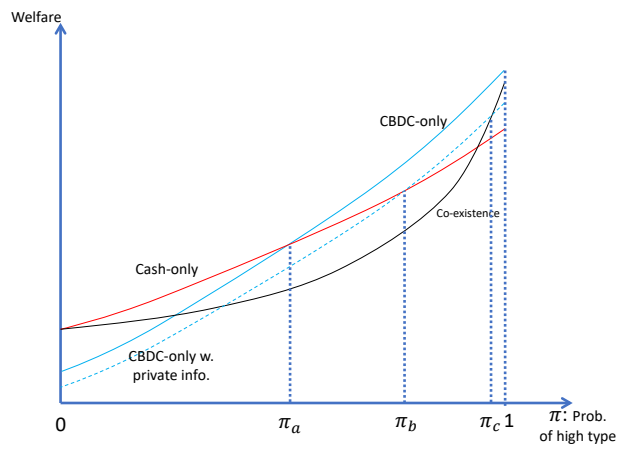

Figure 4: CBDC-only scheme is less likely to be optimal with privacy concerns.

to $\min\left\{(1-\pi)f_2(D_2^{-1}(D_1(q_1^*))) + \pi f_2(q_2^*), f_1(q_1^*)\right\} = \min\left\{(1-\pi)f_2(q_1^*) + \pi f_2(q_2^*), f_1(q_1^*)\right\} = f_1(q_1^*)$. It is shown that in this case, the IC_{12} constraint is slack. In other words, type 1 agents would like to bring more balances to claim to be type 1, but type 2 agents receive a strictly higher payoff compared with the case that they claim to be type 1.

Given the findings from parts (i)-(iii), if the CBDC-only scheme is optimal with complete information for some π and not optimal for other π, as demonstrated in Figure 4, then welfare under the CBDC-only scheme is lower with private information compared with complete information, so the optimal scheme will be different with private information for some π. In Figure 4, the optimal scheme switches to the cash-only scheme for π between π_a and π_b and switches to the co-existence scheme for π between π_c and 1. I summarize this discussion in the following corollary.

Corollary 1. *Suppose* (14) *holds and θ is sufficiently close to 1. Assume that, with complete information, the CBDC-only scheme is optimal for some π and not optimal for other π. Then, the CBDC-only scheme is less likely to be optimal with private information relative to complete information. That is, there exists some π under which the CBDC-only scheme is optimal with complete information but not optimal with private information.*

This corollary is stated in terms of π but can be stated in a similar manner for other parameters. Intuitively, private information in this economy limits the benefits of CBDC. This means that **removal of cash is less likely to be optimal when privacy of transactions from the planner is a concern for the agents.**

7 Calibration

The goal of this section is to calibrate the model to the Canadian and US data to estimate the potential welfare gains of introducing CBDC. I proceed in two steps. First, I estimate the parameters of the model and calculate the welfare costs of inflation. The main parameters to be estimated are the buyer's bargaining power, θ, the elasticity of the utility function, $1-\eta$, where $u(q) = q^{1-\eta}$, and the level of production in the the CM, A. Calculating the welfare costs of inflation is independently interesting, because it allows us to compare the results of the present paper with the existing literature using US data. Second, I use the parameters to find potential gains of introducing CBDC. I also characterize which one of the three schemes, cash-only, CBDC-only or co-existence, is optimal given different parameters of the model.

The data used in the analysis are M1, gross domestic product (GDP) in market prices and the three-month interest rate. For Canada, the data are from CANSIM for the period 1967-2008. For the US, the data are taken from Craig and Rocheteau (2006) for the period 1900-2000. For the sake of brevity, more details on the data sources, the estimation procedure and parameter restrictions come in the appendix.

The estimation method used here is mostly based on Lucas (2001). The theory shows that money demand, M/P, is proportional to real output, Y. Their ratio, $M/(PY)$, is a decreasing function of the opportunity cost of holding non-interest-bearing means of payments; i.e., the nominal interest rate. One way to calculate the welfare costs of inflation is to estimate this function from the data by finding the best fit, then calculate the area under the demand curve from the inflation level of π_0 to $\pi_0 + 0.10$ to estimate the welfare costs of 10% inflation. Another way is to use a model that generates a money demand function. One should then try to find the parameters of the model such that the money demand function generated by the model fits the data as much as possible. Finally, one can calculate the welfare costs of inflation implied by the model. Many papers, including Lucas (2001), Lagos and Wright (2005) and Craig and Rocheteau (2008), also use the latter approach to estimate welfare costs of inflation for the US data. I follow this methodology.

In the first step, I estimate parameters (θ, η, A), taking as given the discount factor, β, and the probability of matching in the DM, σ. Throughout this section, I set $\beta = 0.97$ following Craig and Rocheteau (2008); that is, the real interest rate is set to 3%. I assume that $c(q) = q$; that is, the cost of producing q units of the DM good for sellers is q. In the benchmark exercise, I set $\sigma = 0.5$, but I estimate parameters assuming $\sigma = 1$ as well.[22]

[22] I fix σ in the estimation. If I include σ in the optimization parameters, the estimates of welfare costs

For estimating parameters, I take the fraction of low-value $(1-\pi)$ to high-value (π) transactions to match the ratio of volume of cash transactions to debit card transactions. This data is taken from the Payments Canada report (2018) for Canada and from the 2017 Survey of Consumer Payment Choice (SCPC) by Greene and Stavins (2018) for the US. For Canada, $\pi = 58B/(65B + 58B) = 0.471$, where the total volume of transactions in cash and debit is 65 and 58 billion transactions. For the US, $\pi = 10.7/(12.4 + 10.7) = 0.464$, where the average number of transactions per month for cash and debit is 12.4 and 10.7.

To pin down w_1 and w_2, I calibrate the ratio of w_2/w_1 such that the relative value of w_2- to w_1-transactions in the model matches the ratio of average value of debit transactions to cash transactions in the data. For Canada, $z_2/z_1 = \$CA42.59/\$CA17.54 = 2.43$. For the US, $z_2/z_1 = \$US47/\$US23.40 = 2.01$. The denominator and numerator represent the average value of cash and debit transactions for each country. For the results to be comparable with the aforementioned papers above, I normalize w_1 and w_2 such that $\pi_1 w_1 + \pi w_2 = 1/(1-\eta)$. That is, the average buyer's utility from consuming q units of the DM good is $q^{1-\eta}/(1-\eta)$.

With a linear production function in the the CM, the level of production in equilibrium is indeterminate. Following the literature, I assume the production function in the the CM is $U(X) = A\ ln(X)$. This implies that the level of production in the the CM is $X^* = A$.

I estimate (θ, η, A) by minimizing the distance between the data and the model-generated real balances–income ratio, $M/(PY)$, subject to the constraint that the markup (price over marginal cost minus 1) in the DM is $\mu = 20\%$ under the 2% inflation rate in the benchmark estimation.[23] Fixing the markup imposes a constraint jointly on θ and η.

In the estimation, I assume that only cash is available. I use $M1$ in the data to represent cash in the model. A defining feature of cash in my model is that it cannot bear interest. Similarly, elements of $M1$ are either not interest bearing (like currency) or have historically interest rates close to zero (such as demand deposits whose interest rates were zero due to regulation Q in the US). Introducing CBDC is then equivalent to introducing an interest-bearing money into the economy. For the counter-factual exercise of introducing CBDC, in the first scenario, I assume that all high-value transactions will be conducted by CBDC and report the results based on that. In the real world, most non-cash money today is in the form of deposits at commercial banks, so one may argue that this scenario is not likely to realize because banks would endogenously respond to keep the deposits, so CBDC cannot

of inflation do not change substantially, but the estimates of parameters (θ, η) would be very sensitive, in that various pairs of (θ, η) lead to almost the same fit. Yet, I conduct various robustness checks depicted in Tables 1 and 2.

[23] For the robustness check, I consider a higher value for μ as well. Also, I consider the average markup of both DM and the CM and estimate parameters, with this average markup being 10%.

capture all high-value transactions. Even if banks respond, as long as they pay interest on the deposits to keep them in their banks, the gains from an interest-bearing CBDC will be realized, so my estimates would not be affected.[24] Yet, banks offer various services such as managing customers' accounts and providing them with monthly statements. By offering these services, banks will continue to keep a fraction of deposits even if they do not offer an attractive interest rate on them. For this reason, I report my results under a low-adoption scenario—a pessimistic scenario in terms of CDBC adoption—where 80% of buyers in the model do not use CBDC regardless of the transfers that it may offer. This would give a conservative estimate on the potential gains of an interest-bearing CBDC.[25]

The estimates for Canada and the US are reported in Tables 1 and 2, respectively. The benchmark estimates are shown in the top row of the tables in bold. The real balances–income ratio is plotted against the nominal interest rate for both data points and model-generated points based on the benchmark estimates in Figures 5 and 7 for Canada and the US, respectively. The welfare costs of inflation based on the benchmark estimates are shown in Figures 6 and 8. Following Lagos and Wright (2005), the welfare costs of a given level of inflation are calculated as the fraction of consumption that agents are willing to forgo to be in equilibrium with a 0 inflation rate (or 3% interest rate).

The welfare cost of 10% inflation in the benchmark estimation is 0.92% for Canada and 1.71% for the US. My estimates of welfare costs of 10% of inflation for the US are close to estimates in the literature. Lagos and Wright (2005) estimate these welfare costs as ranging from 1% to 5% and Craig and Rocheteau (2008) estimate these costs as ranging from 0.5% to 5%. Also, my estimate is close to the upper bound in the estimate of Lucas (2001) (less than 1%).

In the second part, I estimate the welfare gains of introducing CBDC. See the last four columns of Tables 1 and 2. These estimates crucially depend on the choice of the cost of

[24] See Keister et al. (2019) for a model in which perfectly competitive banks offer interest-bearing deposits that are used for payments. They study the effects of introducing an interest-bearing CBDC. Chiu et al. (2019) extend that analysis to the case where banks have market power. They also estimate that introducing an interest-bearing CBDC can increase output up to 0.5% for the US, although they do not calculate welfare gains nor do they study the optimal policy.

[25] To calculate welfare for this scenario, I divide the population to two groups called *biased*, who do not use CBDC under any circumstances, and *unbiased*, who may use CBDC if it yields them a higher payoff than cash. I assume 80% of w_1 buyers and 80% of w_2 buyers are biased and the rest are unbiased. I again consider three scheme: In the cash-only scheme, all buyers use cash, in the "co-existence" scheme, all buyers use cash except the unbiased w_2 buyers, and in the "CBDC-only" scheme, all biased buyers use cash and unbiased buyers use CBDC. Similar to the main analysis, I calculate welfare gains of CBDC relative to the cash-only scheme.

carrying CBDC relative to cash in a transaction, K.

I calculate a range for possible gains of introducing CBDC when the cost of carrying CBDC in transactions relative to cash ranges from 0% to 0.69% of the transaction value for Canada and from 0% to 0.84% for the US.[26] The interpretation of these upper-bounds is that, if the cost of carrying CBDC exceeds them, the gains of introducing CBDC become zero. The welfare gains of introducing CBDC are calculated relative to 0% inflation. That is, I calculate welfare at 0% inflation when only cash is used, and then compare it with the optimal level of welfare when CBDC can be used too. More precisely, I calculate the level of additional consumption that makes the agents indifferent between being in equilibrium with 0% inflation with only cash circulating in the economy, and being in equilibrium under the optimal policy with both cash and CBDC. When the cost of carrying CBDC relative to cash is 0%, the welfare gains of introducing CBDC are 0.101% for Canada and 0.250% for the US. With other parameter specifications, the welfare gains range from 0.101% to 0.146% for Canada and from 0.250% to 0.344% for the US.

Here are some observations from the estimations other than the benchmark one. I allow the ratio of high-value to low-value transactions to be counter-factually higher (to be equal to 5) in the last row. The welfare loss of 10% inflation does not change with the ratio of high-value to low-value transactions, but the welfare gains of introducing CBDC increase, because the high types are the ones who benefit from CBDC and increasing their share in the population clearly increases the welfare gains. A similar point applies to π. See other rows for different specifications.

Using the parameter estimates from the benchmark estimations, I show the region of parameters under which the cash-only, the CBDC-only or the co-existence schemes are optimal for Canada and the US in Figure 9. Here are the insights obtained from this exercise. As π increases, CBDC is more likely to be used, although not much can be said about how the threshold above which agents use CBDC changes with β. As β increases, cash is more likely to be used, because cash is less costly. The co-existence scheme is optimal only for a small subset of parameter values. As K increases, CBDC is unsurprisingly less likely to be

[26] I have motivated the cost of carrying CBDC as the value of anonymous transactions that the agents would lose if they use CBDC instead of cash. Further research is needed to accurately estimate this value. However, according to a work-in-progress by Jiaqi Li, who uses Means of Payments survey for Canadian data to study the household's portfolio choice, the deposit to cash ratio for the median household remains identical if the household takes either of the following portfolio choices: one asset with a given interest rate and non-anonymous transactions, and another asset with 0.16% lower interest rate but with anonymous transactions. This is a suggestive evidence that the value of anonymous transactions for agents is in the range of 0.16%. If so, CBDC would bring welfare gains according to my estimates.

used, but surprisingly, the co-existence is less likely to be optimal. This is because, quantitatively, the effect of a tighter IC constraint under the co-existence scheme is dominant. These insights are robust across alternative calibration parameters.

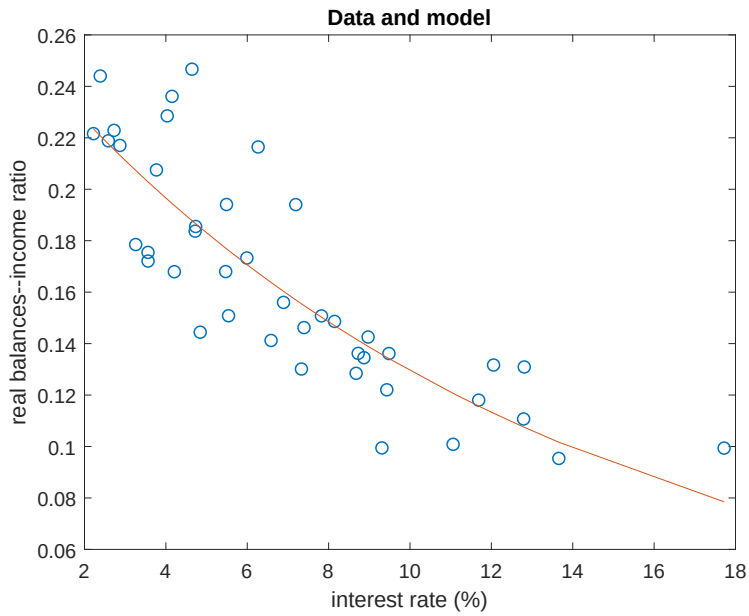

Figure 5: Data and model for Canada

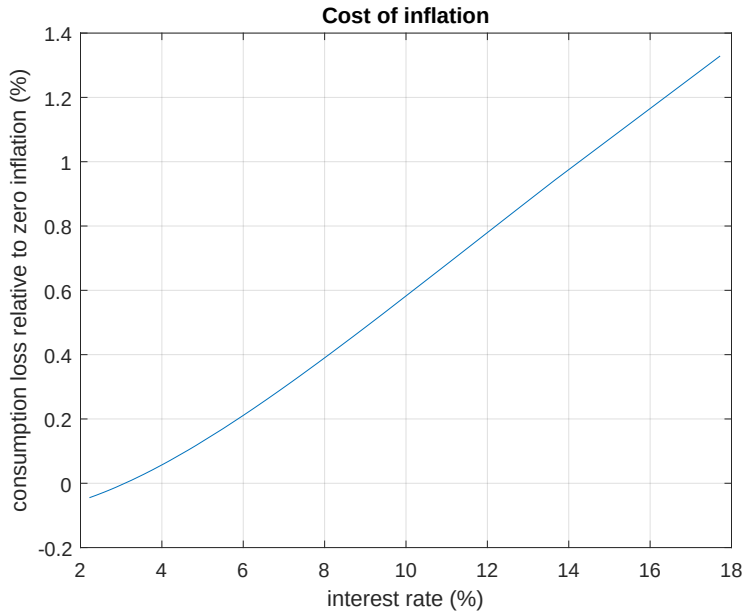

Figure 6: Welfare cost of inflation for Canada

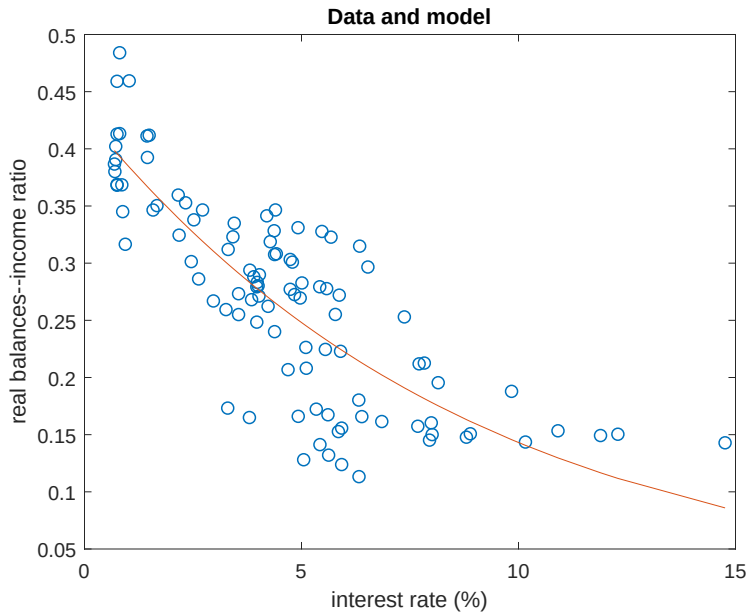

Figure 7: Data and model for the US

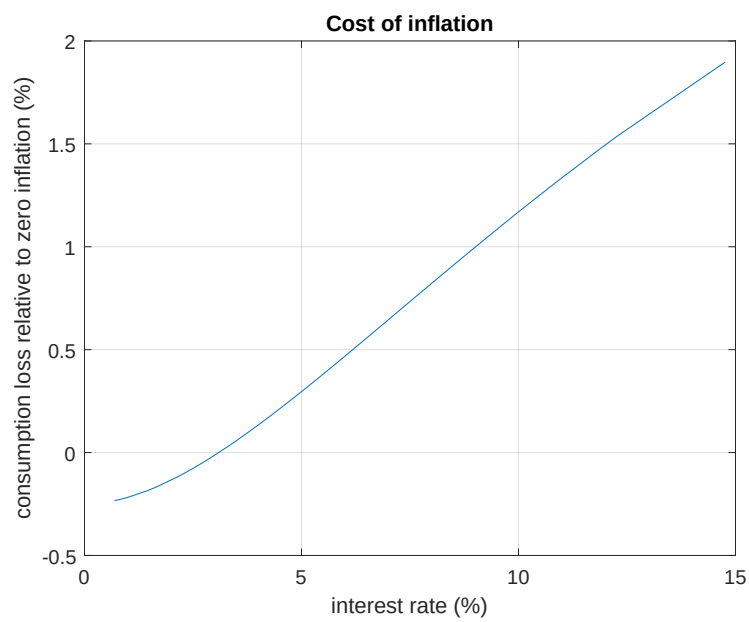

Figure 8: Welfare cost of inflation for the US

π	w_1	w_2	σ	θ	η	A	$\frac{z_2}{z_1}$	Welfare loss of 10% inflation	μ	CBDC cost as a fraction of average transaction value (%)	Who uses CBDC?	Welfare gains of CBDC (%)	Welfare gains of CBDC (%); low adoption
										0.000	**both**	**0.101**	**0.016**
										0.224	both	0.039	0.004
0.472	**1.200**	**1.507**	**0.500**	**0.883**	**0.256**	**3.692**	**2.428**	**0.918**	**0.200**	0.326	w_2	0.021	0.001
										0.486	w_2	0.000	-0.003
										0.693	**none**	**-0.014**	**-0.006**
										0.000	both	0.111	0.018
0.472	1.223	1.565	0.500	0.797	0.278	3.830	2.428	1.023	0.400	0.260	both	0.039	0.004
										0.669	none	-0.006	-0.005
										0.000	both	0.112	0.017
0.472	1.087	1.220	1.000	0.859	0.130	3.218	2.428	1.004	0.200	0.270	both	0.034	0.003
										0.706	none	-0.012	-0.006
										0.000	both	0.146	0.022
0.472	1.102	1.268	1.000	0.699	0.150	3.217	2.428	1.245	0.100*	0.262	both	0.069	0.006
										0.589	w_2	0.007	-0.004
										0.000	both	0.103	0.016
0.800	1.117	1.402	0.500	0.883	0.256	3.567	2.428	0.918	0.200	0.287	w_2	0.026	0.000
										0.718	none	-0.039	-0.015
										0.000	both	0.105	0.016
0.472	1.084	1.637	0.500	0.883	0.256	4.276	5.000	0.918	0.200	0.246	w_2	0.048	0.006
										0.493	w_2	0.014	-0.002

Table 1: Estimated parameters and estimates of welfare gains of introducing CBDC for Canada. The gains are calculated relative to an economy with only cash under 0% inflation. *: the average markup in both CM and DM.

π	w_1	w_2	σ	θ	η	A	$\frac{z_2}{z_1}$	Welfare loss of 10% inflation	μ	CBDC cost as a fraction of average transaction value (%)	Who uses CBDC?	Welfare gains of CBDC (%)	Welfare gains of CBDC (%); low adoption
										0.000	**both**	**0.250**	**0.036**
0.464	**1.132**	**1.269**	**0.500**	**0.870**	**0.163**	**1.972**	**2.009**	**1.706**	**0.200**	0.278	both	0.121	0.005
										0.841	**none**	**-0.008**	**-0.012**
										0.000	both	0.302	0.042
0.464	1.054	1.109	1.000	0.851	0.074	1.483	2.009	1.896	0.200	0.280	both	0.164	0.007
										0.561	both	0.027	-0.013
										0.000	both	0.323	0.045
0.464	1.058	1.118	1.000	0.794	0.079	1.491	2.009	2.039	0.300	0.279	both	0.186	0.008
										0.558	both	0.049	-0.012
										0.000	both	0.344	0.048
0.464	1.062	1.126	1.000	0.745	0.084	1.502	2.009	2.181	0.100*	0.277	both	0.208	0.010
										0.555	both	0.072	-0.011
										0.000	both	0.257	0.036
0.800	1.090	1.222	0.500	0.870	0.163	1.927	2.009	1.706	0.200	0.285	both	0.121	0.002
										0.861	none	-0.015	-0.027
										0.000	both	0.269	0.036
0.464	1.049	1.365	0.500	0.870	0.163	2.408	5.000	1.706	0.200	0.228	both	0.155	0.015
										0.511	w_2	0.082	0.001

Table 2: Estimated parameters and estimates of welfare gains of introducing CBDC for the US. The gains are calculated relative to an economy with only cash under 0% inflation. *: the average markup in both CM and DM.

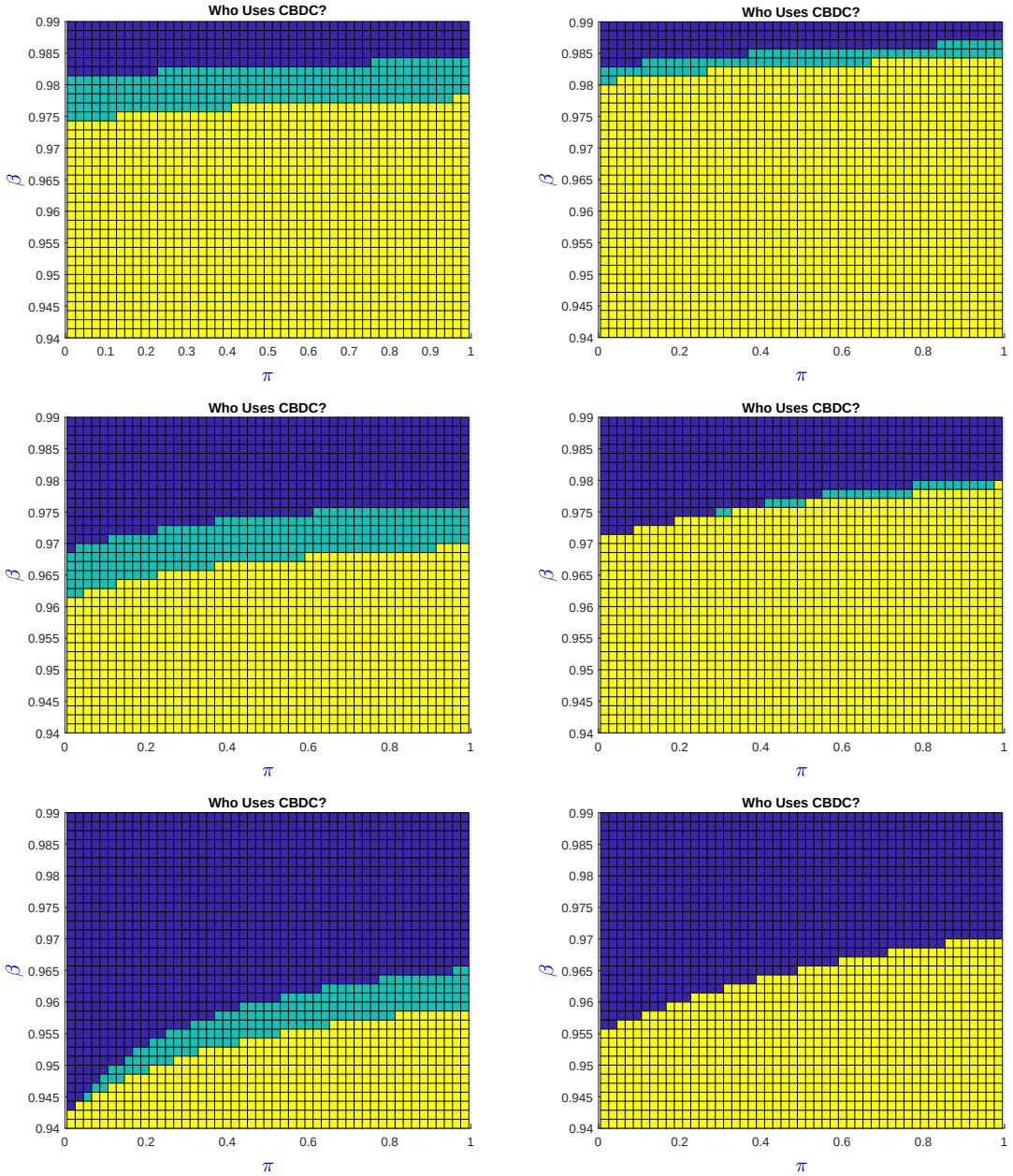

Figure 9: Who uses CBDC under the optimal scheme?

Note: The graphs are depicted for Canada (left) and the US (right) for different values of π and β for $K = 0.002$ (top), $K = 0.004$ (middle) and $K = 0.008$ (bottom)? Yellow: Both types use CBDC; Green: only high type uses CBDC; Blue: no type uses CBDC.

8 Extension

I assume in this section that there is no direct cost of carrying CBDC—i.e., $K = 0$—but CBDC cannot be used in a fraction of transactions. For example, the seller may not be able to accept CBDC because no internet or electricity connection is available, or because the seller does not have access to CBDC technology (such as un-banked agents in remote locations). Another reason might be that the buyers do not want to use CBDC in some transactions so that they can keep their anonymity. Formally, assume that in $\alpha_c \in [0, 1]$ fraction of transactions, only cash can be used (c-meetings); in $\alpha_e \in [0, 1]$ fraction of transactions, only CBDC can be used (e-meetings); and in $\alpha_b \in [0, 1]$ fraction of transactions, both cash and CBDC can be used (b-meetings).[27] Of course, $\alpha_c + \alpha_e + \alpha_b \leq 1$.[28] Note that the buyers learn the type of the meeting only after they match with a seller. Therefore, and in contrast to the previous sections, a given agent may want to hold both means of payment as they do not know which means of payment can be used in the DM. The case of $\alpha_c = 0$ is similar to the benchmark model in the previous sections with $K = 0$, in that carrying CBDC is costless, so cash is redundant. In the case of $\alpha_c > 0$, CBDC users may effectively incur a cost for using CBDC, as their means of payment cannot be used in some transactions.

I assume without loss of generality that the planner is not allowed to make cash transfers, i.e., $t_c = 0$.[29] Furthermore, I do not study the heterogeneity of types in this section; type w is fixed and identical for all buyers.[30] Therefore, the index w is removed from the notation. The rest of the environment is identical to the benchmark model. In particular, I continue to assume that the transfer to agents cannot be negative.

For a given policy, the agent's problem is given as follows:

$$\max_{z_c, z_e} \Big\{ -(\gamma_c - \beta) z_c - (\gamma_e - \beta) z_e + \beta t_e(z_e) + \beta\sigma\theta \Big(\alpha_c f(q_c) + \alpha_e f(q_e) + \alpha_b f(q_b) \Big) \Big\}$$

[27] In making this assumption, I follow Rocheteau et al. (2014) and Zhu and Hendry (2017). In the former, fiat money and bond, and in the latter, money and private e-money are means of payment.

[28] Especially when doing comparative statics, I allow their sum to be less than 1. The interpretation is that in some transactions, the buyer and seller may not trade with any means of payment.

[29] To show that this is without loss of generality, start from an allocation with a strictly positive t_c. Change t_c to 0 but conduct an OMO such that the same amount of cash is injected into the economy in exchange for CBDC. Next, transfer CBDC to the agents (and possibly adjust the CBDC inflation) so that the DM production levels in all three types of meetings remain the same.

[30] I have heterogeneity of types in the benchmark model to generate demand for both means of payment endogenously. In this section, since one means of payment may not be useful in some transactions, agents may want to bring both means of payment from the CM, so heterogeneity is not needed.

where

$$q_c = \min\{q^*, D^{-1}(z_c)\}, \tag{15}$$

$$q_e = \min\{q^*, D^{-1}(z_e + t_e)\}, \tag{16}$$

$$q_b = \min\{q^*, D^{-1}(z_c + z_e + t_e)\}. \tag{17}$$

The planner's problem is to choose a policy that maximizes welfare, and can be written as follows:

$$\max_{z_c, z_e, t_e(.), \gamma_c, \gamma_e} \{\alpha_c f(q_c) + \alpha_e f(q_e) + \alpha_b f(q_b)\}$$
$$\text{s.t. } t_e(z_e) = (\gamma_c - 1)z_c + (\gamma_e - 1)z_e, \tag{18}$$

where (q_c, q_e, q_b) is obtained from the agent's maximization problem given above.

Similar to the benchmark model, we can assume without loss of generality that if an agent brings CBDC, then the agent will bring the exact amount of CBDC that the planner asks for. To support this punishment, the inflation rate for CBDC should be sufficiently high that the CBDC transfers for agents who bring enough balances become positive. Therefore, the only relevant constraint is that agents should not gain if they bring all their portfolio in the form of cash. This constraint is summarized by

$$\max_{z_c} \big\{ -(\gamma_c - \beta)z_c + \beta\sigma\theta\big(\alpha_c f(q_c) + \alpha_b f(q_b)\big)\big\} - (\gamma_e - \beta)z_e + \beta t_e(z_e) + \beta\sigma\theta\alpha_e f(q_e)$$
$$\geq \max_{q} \big\{ -(\gamma_c - \beta)D(q) + \beta\sigma\theta(\alpha_c + \alpha_b)f(q)\big\}. \tag{19}$$

Also, define

$$z_c \in \arg\max_{z_c'} \big\{-(\gamma_c-\beta)z_c'+\beta\sigma\theta\big(\alpha_c f(\min\{q^*, D^{-1}(z_c')\})+\alpha_b f(\min\{q^*, D^{-1}(z_c'+z_e+t_e(z_e))\})\big)\big\}. \tag{20}$$

From now on, I summarize the optimal policy by z_c and q_b. From (15), z_c pins down q_c. One then obtains γ_c from (20) consistent with z_c, q_b and q_c, and obtains z_e and t_e from (17) and (18) for an arbitrarily large γ_e. Finally, q_e is given by (16).

8.1 Sufficient Conditions to Achieve the First Best

Remember that in Section 4, the first-best level of welfare cannot be achieved when $K > 0$, because even if β and θ are sufficiently high, the buyers still have to incur the direct cost of carrying CBDC. In this section, in contrast, **the first-best level of welfare can be achieved even when $\alpha_c > 0$, if β and θ are sufficiently high**. Notice that $\alpha_c > 0$ is a reason that the agents prefer cash over CBDC, the same role that $K > 0$ plays in the

benchmark model. Hence, the main message from this result is that assuming a reduced form cost for carrying CBDC (assuming $K > 0$) leads to a different result than explicitly modeling the underlying reason for agents to prefer cash over CBDC (assuming $\alpha_c > 0$).

Proposition 11. *Assume $\alpha_c = 0$. Then the first best is achievable if and only if*

$$\beta\sigma(\alpha_e + \alpha_b)\theta(wu(q^*) - c(q^*)) \geq (1-\beta)D(q^*). \tag{21}$$

Now assume that $\alpha_c > 0$. The first best is achievable if

$$\beta\sigma\alpha_e\theta(wu(q^*) - c(q^*)) \geq 2(1-\beta)D(q^*). \tag{22}$$

If $\alpha_c = 0$, the condition needed to achieve the first best is equivalent to that in Proposition 3. In the general case of $\alpha_c > 0$, to achieve the first best, the planner needs to ensure that buyers have enough balances in all meetings to achieve the first best. Regarding the c-meetings, the planner runs deflation on cash at the rate of time preferences—i.e., $\gamma_c = \beta$. Regarding the e-meetings, the planner simply needs to ensure that the number of balances in CBDC that the agents carry, together with what they receive as transfers, is enough to buy the first-best level of production. The planner runs inflation on CBDC to finance the cash deflation through an OMO (by withdrawing cash and injecting CBDC). Similar to the transfer scheme used previously, the planner makes CBDC transfers to agents only if they bring enough CBDC balances from the CM. The planner would not worry about the b-meetings because agents have enough balances to buy the first-best level of consumption by using only one means of payment.

8.2 Optimal Scheme When the First Best Cannot Be Achieved

Now assume that the first best is achievable when there are no c-meetings—i.e., when $\alpha_c = 0$—but it is not achievable otherwise. That is, (21) holds but (22) does not. We are interested in knowing whether both means of payment are valued in the optimal allocation (co-existence) or only one means of payment is valued. Particularly, how is the use of means of payment changed when α_c goes to zero (even when Inada condition is not satisfied)? Do buyers choose to bring a small amount of cash from the CM? Or, is there a threshold for α_c below which they do not use cash even though cash is useful in some meetings? This exercise is useful, because cash usage in advanced economies is in decline. It is interesting to see whether the optimal policy for the planner is to help this down-ward trend and remove cash as early as possible, or maintain and potentially promote cash usage along the transition path.

Proposition 12. *Suppose $\alpha_c + \alpha_e + \alpha_b = 1$ and that $\alpha_c > 0$. Assume that the first best can be achieved when $\alpha_c = 0$ (no c-meetings) but not when $\alpha_c > 0$. Then, the following holds:*

(a) Assume $\alpha_b = 0$. Suppose (21) holds with strict inequality. Then, cash is valued in the optimal allocation.

(b) Assume $\alpha_e = 0$. If both means of payment are valued and the first best cannot be achieved in any meetings, then cash inflation must be strictly positive.

Part (a) states that when $\alpha_b = 0$ and $\alpha_c > 0$, it is optimal to use cash, even without Inada conditions. Since there are no b-meetings, cash is not a substitute for CBDC. Therefore, effectively, the incentive constraint does not exist and cash is valued as long as there are some meetings for which cash is useful. That is, **as the economy is going cashless, it is optimal to keep cash.** This result is in contrast to the result in the benchmark model in which under certain conditions (e.g., small K with homogeneous buyers or requirements of Proposition 9 with heterogeneous buyers), cash is not used in the optimal allocation.

Part (b) simply states that cash inflation is necessary to provide incentive to agents to hold CBDC. If cash inflation is too low, agents would have incentive to use cash in both c-meetings and b-meetings, and therefore, the welfare gains associated with using CBDC would not be realized. This result is similar to the benchmark model in Proposition 8. The tradeoff that the planner faces is similar to that in the benchmark model. If cash is valued in the equilibrium, then agents can substitute cash for CBDC in the b-meetings. Therefore, agents have less incentive to bring CBDC. This implies that welfare gains from CBDC, due to its flexibility, are less likely to be realized. If cash is not valued, there will be no trade in the c-meetings. The optimal policy trades off these two forces.

The following proposition further characterizes which scheme, cash-only, CBDC-only or co-existence, is optimal.

Proposition 13. *(a) Assume $\alpha_e = 0$. There exists $\bar{\alpha}_c < 1$ such that if $\alpha_c \geq \bar{\alpha}_c$, then only cash is valued in the optimal allocation.*

(b) Assume $\alpha_b = 0$ and $u'(0) < \infty$. There exists $\tilde{\alpha}_c < 1$ such that if $\alpha_c \geq \tilde{\alpha}_c$, then only cash is valued in the optimal allocation.

I summarize the analytical results so far in the space of α_c and α_e in Figure 10, in which I assume $\alpha_c + \alpha_e + \alpha_b = 1$. Ideally, one would like to identify the regions under which each scheme is optimal, but it is analytically hard to do that for the entire space, so I identify the optimal scheme only on some boundaries of the region. When $\alpha_c = 0$, cash is redundant, and if (21) holds, the first best is achievable. If $\alpha_e = 0$, then Proposition 13(a) implies that the cash-only scheme is optimal. Similarly, if $\alpha_b = 0$, then the cash-only scheme is optimal from

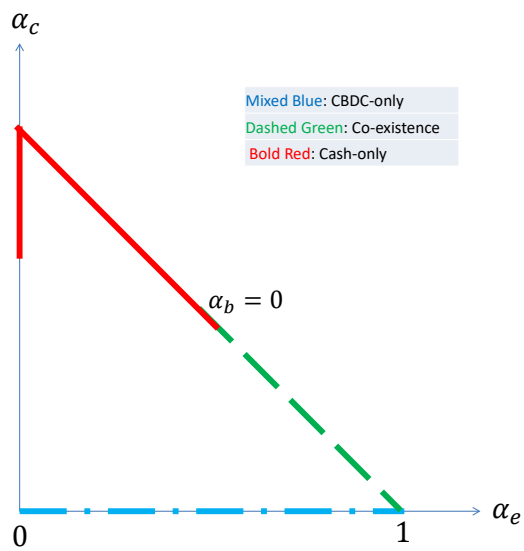

Figure 10: Optimal scheme

Note: The results regarding the optimal means of payment are demonstrated here, where $\alpha_b = 1 - \alpha_c - \alpha_e$. The blue line corresponds to Proposition 11. The green line corresponds to Proposition 12(a) and the red lines correspond to Proposition 13(a) and (b).

Proposition 13(b). If $\alpha_b = 0$, then Proposition 12(a) implies that cash should be valued. Also, Proposition 11 implies that for a sufficiently large α_e, CBDC should be valued, too.

An interesting observation is the asymmetry between usage of cash and CBDC. Assume $\alpha_b = 0$. If α_c is sufficiently close to 1, then using CBDC is not optimal (Proposition 13(b)), but if α_c is sufficiently close to 0, using cash is still optimal (Proposition 12(a)), because the taxation policy associated with CBDC can help achieve better allocations even in the c-meetings.

9 Concluding Remarks

In this paper, I study a model in which agents can hold two central bank-issued monies: cash and CBDC. CBDC is interest bearing potentially in a non-linear fashion, but at the same time, there is a cost associated with carrying it, perhaps due to the agents' anonymity concerns. I characterize the welfare-maximizing monetary policy with respect to these two means of payment. Assume first that cash and CBDC are perfect substitutes in the transactions. If the cost of carrying CBDC is small enough, then only CBDC is used under the optimal policy, and even the first best can be achieved. If CBDC is too costly, only cash

is used. If the cost of carrying CBDC is intermediate, welfare under co-existence may be dominated by the scheme in which only one means of payment is used. This is because under co-existence, agents use cash as a way to evade the taxation scheme that CBDC users are subjected to; therefore, co-existence may lead to under-utilization of CBDC, which has an attractive feature of bearing interest. When cash and CBDC are not perfect substitutes, then co-existence is more likely to be optimal. Furthermore, CBDC can help achieve better allocations even for the meetings in which only cash can be used. This is possible through an OMO, where the planner exchanges cash for CBDC in order to run the Friedman rule for cash.

A system that can implement the CBDC discussed in this paper is a debit card system that is owned and monitored by the central bank (although its operations can be outsourced to other institutions such as "FinTech" companies, if they can implement it with low operational costs). Each individual can have an account with the central bank, and individuals can use these balances for purchases of goods and services. Such a system provides access to the central bank balance sheet in electronic format for all agents in the population and allows them to earn interest on their balances. Currently, only some financial institutions have this privilege. With such a system, monetary policy directly affects agents' decisions to carry balances, rather than through the financial system, making the implementation of monetary policy more transparent. (See Berentsen and Schar (2018).)

References

R. Agarwal and M. Kimball. Breaking through the zero lower bound. *IMF Working Paper No. 15/224*, 2015.

D. Andolfatto. Essential interest-bearing money. *Journal of Economic theory*, 145(4):1495–1507, 2010.

J. Barrdear and M. Kumhof. The macroeconomics of central bank issued digital currencies. *Bank of England Staff Working Paper No. 605*, 2016.

M. Bech and R. Garratt. Central bank cryptocurrencies. *BIS Quarterly Review*, page 55, 2017.

A. Berentsen and F. Schar. The case for central bank electronic money and the non-case for central bank cryptocurrencies. *Federal Reserve Bank of St. Louis Review, Early Edition*, 2018.

O. Bjerg. Designing new money-the policy trilemma of central bank digital currency. *Copenhagen Business School Working Paper*, 2017.

J. Chiu and T.-N. Wong. On the essentiality of e-money. Technical report, Bank of Canada Staff Working Paper, 2015.

J. Chiu, M. Dong, and E. Shao. On the welfare effects of credit arrangements. *Bank of Canada Staff Working Paper 2012-43*, 2012.

J. Chiu, S. M. Davoodalhosseini, J. Hua Jiang, and Y. Zhu. Central bank digital currency and banking. *Bank of Canada Staff Working Paper*, (No. 2019-20), 2019.

Committee on Payments and Market Infrastructures. Digital currencies. *Bank of International Settlements*, November 2015.

B. Craig and G. Rocheteau. Inflation and welfare: A search approach. *Federal Reserve Bank of Cleveland Policy Discussion Paper Number 12*, 2006.

B. Craig and G. Rocheteau. Inflation and welfare: A search approach. *Journal of Money, Credit and Banking*, 40(1):89–119, 2008.

M. Davoodalhosseini and F. Rivadeneyra. A policy framework for e-money. *Canadian Public Policy*, 46(1):94–106, 2020.

M. Dong and J. H. Jiang. One or two monies? *Journal of Monetary Economics*, 57(4): 439–450, 2010.

J. Fernández-Villaverde and D. Sanches. Can currency competition work? Technical report, National Bureau of Economic Research, 2016.

B. Fung and H. Halaburda. Central bank digital currencies: a framework for assessing why and how. *Bank of Canada Staff Discussion Paper 2016-22*, 2016.

B. Fung, K. P. Huynh, and G. Stuber. The use of cash in canada. *Bank of Canada Review*, 2015(Spring):45–56, 2015.

P. Gomis-Porqueras and D. Sanches. Optimal monetary policy in a model of money and credit. *Journal of Money, Credit and Banking*, 45(4):701–730, 2013.

C. Greene and J. Stavins. The 2017 diary of consumer payment choice. *Discussion Paper*, 30(18-05), 2018.

C. Gu, F. Mattesini, and R. Wright. Money and credit redux. *Econometrica: Journal of the Econometric Society*, 84(1):1–32, 2016.

C. Kahn, F. Rivadeneyra, and T.-N. Wong. E-money and payment policy. Technical report, Bank of Canada Staff Working Paper, 2017.

T. Keister, D. Sanches, et al. Should central banks issue digital currency? *Federal Reserve Bank of Philadelphia Working Paper*, (No. 19-26), 2019.

R. Lagos and R. Wright. A unified framework for monetary theory and policy analysis. *Journal of Political Economy*, 113(3):463–484, 2005.

R. E. Lucas. Inflation and welfare. In *Monetary Theory as a Basis for Monetary Policy*, pages 96–142. Springer, 2001.

Payments Canada report. Canadian payment methods and trends: 2018. 2018.

G. Rocheteau and R. Wright. Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium. *Econometrica*, 73(1):175–202, 2005.

G. Rocheteau, R. Wright, and S. X. Xiao. Open market operations. Technical report, Mimeo, 2014.

K. S. Rogoff. *The curse of cash*. Princeton: Princeton University Press, 2016.

C. Skingsley. Should the riksbank issue e-krona? *Speech at FinTech Stockholm*, 16 November, 2016.

Sveriges Riksbank. The riksbank's e-krona project: Report 1. September 2017.

Sveriges Riksbank. The riksbank's e-krona project: Report 2. October 2018.

J. Tobin. A case for preserving regulatory distinctions. *Challenge*, 30(5):10–17, 1987.

Y. Zhu and S. Hendry. A framework for analyzing monetary policy with e-money. Technical report, Mimeo, 2017.

Appendix

In this appendix, I first discuss the relationship between my paper and two closely related papers. Second, I provide proofs for the results. Finally, I elaborate on the calibration exercise.

Relation with Chiu and Wong (2015)

My paper is among the first to study the payment choice between cash and CBDC, and how this choice affects the optimal monetary policy. The only paper with something of a similar objective is Chiu and Wong (2015). Perhaps the most important difference between my model and theirs is that the features of cash and CBDC emphasized in my model are different from theirs. In particular, I do not allow for communication between the planner and the agents if they use cash. Also, I do not allow for the planner's direct interventions in the bilateral meetings. I believe these two features are more realistic regarding cash and the versions of CBDC typically discussed in policy circles.

To elaborate, the first difference between my paper and theirs is with respect to how cash transfers are made to agents. In their model, the agents can communicate with the planner or mechanism designer to report their cash holdings, and the planner proposes an allocation that can depend on their report in a very general way; i.e., there is no requirement on the functional form of this dependence. In my model, the ability of the planner to implement monetary policy around cash is very limited. No communication between cash users and the planner is allowed, and the only possible form of implementing monetary policy regarding cash is the helicopter drop; i.e., the number of cash transfers to agents cannot depend on the agent's characteristics (such as transaction amount). As a result, the first best is never achievable with cash in my model. Roughly speaking, the CBDC in my paper is equivalent to cash, together with some form of communication in their paper.

The second difference is that in my model, the planner cannot intervene in the bilateral meetings. All interventions in my model are done either in the CM (using OMO) or in the transfer stage before bilateral meetings occur. In contrast, in their model, the planner has the power to either restrict access to CBDC balances in the DM or make transfers to the agents in the DM. Therefore, the set of interventions available to the planner in their paper is different from my paper.[31]

[31] There are other differences between the two papers, too. In my paper, the agents are heterogeneous in

Relation with Gomis-Porqueras and Sanches (2013)

I do not assume that the planner has access to the history of transactions, but the fact that the planner can see the current CBDC balances somewhat summarizes all available information for the planner whether the agent has worked in the previous CM or not. In this sense, CBDC in my model acts like credit, which is the focus of their paper. Unlike in my benchmark model, the credit is not available in their paper in a fraction of meetings, so their setting is similar to the setting in Section 7 of my paper. Moreover, they do not have heterogeneity of buyers; therefore, the endogenous adoption of different means of payment for different agents does not happen in their equilibrium. Finally, they do not have a policy choice regarding credit. In contrast, in my model, the policy choice of the planner is to choose how CBDC transfers should be distributed to the agents to induce them to use the means of payment that leads to the socially optimal outcome.

Proofs

Proof of Lemma 1. One can write

$$t_c + \int t_e(\hat{z}_e(w), w)dF(w) = \phi_+ M_+ + \psi_+ E_+ - \phi_+ \bar{M} - \psi_+ \bar{E} = \phi_+(M_+ + \frac{\psi_+}{\phi_+}E_+) - \phi_+(\bar{M} + \frac{\psi}{\phi}\bar{E})$$

$$= \phi_+(\bar{M}_+ + \frac{\psi_+}{\phi_+}\bar{E}_+) - \phi_+(\bar{M} + \frac{\psi}{\phi}\bar{E}) = \frac{\phi_+}{\phi_{++}}\phi_{++}\bar{M}_+ - \phi_+\bar{M} + \frac{\psi_+}{\psi_{++}}\psi_{++}\bar{E}_+ - \psi_+\bar{E}$$

$$= (\gamma_c - 1)\int \hat{z}_c(w)dF(w) + (\gamma_e - 1)\int \hat{z}_e(w)dF(w).$$

For the first equality, (3) and (4) were used. For the third one, (2) was used for $t+1$. Equations (5) and (6) together with (1) were used for the last equality. □

Proof of Proposition 1. Consider problem 2. Take an optimal policy $\{\gamma_c, \gamma_e, t_c, \{t_e(z,w)\}_w\}$, called the original policy, and denote the equilibrium allocation by $\{(q(w), \hat{z}_c(w), \hat{z}_e(w))\}_w$, called the original allocation. I proceed in three steps:

Step 1. Consider another policy, called policy-2, in which $t_e(z,w)$ is replaced by

$$t'_e(z,w) = \begin{cases} t_e(\hat{z}_e(w), w) & z = \hat{z}_e(w) \\ 0 & z \neq \hat{z}_e(w) \end{cases} \text{ for all } w.$$

their transaction needs, while agents are homogeneous in their paper. I study the cases under which both means of payment co-exist and are used by different agents. In their paper, co-existence is not studied.

It is easy to see that the equilibrium objects of the original policy form an equilibrium under policy-2 as well, because agents would get weakly less if they bring balances different from $\hat{z}_e(w)$.

Step 2. Choose $\gamma_e' > 1$ sufficiently large such that:

$$\sup_{w \in [w_{min}, w_{max}]} \{-(\gamma_e' - \beta) D_w'(q(w)) + \beta\sigma\theta(wu'(q(w)) - c'(q(w)))\} < 0.$$

Such γ' exists because w lies in a compact set and $D_w'(q(w)) > 0$.

Step 3. Define $\hat{z}_e'(w) \equiv \frac{\gamma_e \hat{z}_e(w)}{\gamma_e'}$ and $t_{0,w} \equiv \hat{z}_e(w) + t_e(\hat{z}_e(w), w) - \hat{z}_e'(w)$. Now consider the following policy $\{\gamma_c, \gamma_e', t_c, \{t_e''(z,w)\}_w\}$, called policy-3, where

$$t_e''(z,w) = \begin{cases} t_{0,w} & z \geq \hat{z}_e'(w) \\ 0 & z < \hat{z}_e'(w) \end{cases} \text{ for all } w.$$

In words, we have increased the CBDC inflation from γ_e to γ_e' and changed $\hat{z}_e(w)$ and $t_e(w)$ such that $\hat{z}_e(w) + t_e(w)$ and $\gamma_e \hat{z}_e(w)$ remain constant, which in turn implies that the level of consumption under policy-3 for all types remain constant. Notice that the transfers in this policy to CBDC holders is a step function. I argue that the same consumption and welfare levels under the original policy is achievable under policy-3.

Compared with policy-2, the constraints of the planner's problem continue to hold given the construction of $\hat{z}_e'(w)$ and $t_e''(z,w)$. The level of transfers for $z < \hat{z}_e'(w)$ has not changed in policy-3, so type w buyers do not want to bring CBDC any less than that under policy-2. Now, I calculate how much extra payoff they can receive if they bring a marginally higher number of balances, dz, than $\hat{z}_e'(w)$ under policy-3. That is given by:

$$-(\gamma_e' - \beta) D_w'(q(w)) + \beta\sigma\theta(wu'(q(w)) - c'(q(w))).$$

For derivation, see Problem 2 again. As noted in step 3, γ_e' is sufficiently high such that this expression is negative for all w. Finally, the agents' decision regarding cash does not change either, as cash payments to them are the same as before. This completes the proof. □

Proof of Proposition 2. For any optimal policy and the associated equilibrium allocation, another policy and equilibrium is constructed in which cash is not used and the welfare remains the same. Consider problem 2. Take the optimal policy $\{\gamma_c, \gamma_e, t_c, \{t_e(z,w)\}_w\}$ and denote the equilibrium allocation by $\{(q(w), z_c(w), z_e(w))\}_w$. Consider another policy with $t_c' = 0$, $M_0' = M_0$, $E_0' = E_0$ and

$$t_c' = 0, t_e'(z,w) = \begin{cases} D_w(q(w)) - \frac{D_w(q(w)) + \Lambda_c(w) + \Lambda_e(w)}{\gamma_e'} & z \geq \frac{D_w(q(w)) + \Lambda_c(w) + \Lambda_e(w)}{\gamma_e'} \\ 0 & \text{otherwise} \end{cases}$$

where

$$\Lambda_c(w) \equiv (\gamma_c - 1)(t_c + z_c(w)) - \gamma_c t_c,$$

$$\Lambda_e(w) \equiv (\gamma_e - 1)(t_e(z_e(w), w) + z_e(w)) - \gamma_e t_e(z_e(w), w).$$

Set γ_c' sufficiently high such that buyers do not want to bring any balances from the CM, and γ_e' is set later. Then it is easy to check that with this policy, agents will choose the same $q(w)$ as in the original equilibrium.

To see how the new policy was constructed, note that the buyer's payoff can be written as

$$-(1-\beta)(\underbrace{z_c + z_e(w) + t_c + t_e(z_e(w), w)}_{D_w(q)}) + \beta\sigma\theta(wu(q) - c(q)) - \Lambda_c(w) - \Lambda_e(w)$$

following the first constraint in problem 2. I construct γ_e' and $t_e'(z, w)$ such that the real post-transfer balances of buyers remain the same as in the original allocation—i.e., $t_e'(z_e'(w), w) + z_e'(w) = D_w(q(w))$—and also their payoff remains the same:

$$-\Lambda_c(w) - \Lambda_e(w) = -(\gamma_c' - 1)(t_c' + z_c'(w)) + \gamma_c' t_c' - (\gamma_e' - 1)(t_e'(.) + z_e'(w)) + \gamma_e' t_e'(.).$$

Under this policy, if any buyer wants to carry cash, the buyer gets at most $\max_q\{-(\gamma_c' - \beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\}$, which is 0 if γ_c' is set sufficiently high. If any buyer chooses to bring balances in CBDC but lower than $z_e'(w)$, then the buyer can receive no more than his payoff if he uses cash, in which case his payoff is 0. If the buyer brings $z_e'(w)$, then the buyer receives a positive payoff as in the original equilibrium allocation.

Finally, I show below that the last constraint in the problem is also satisfied.

$$t_c' + \int \left[(t_e'(z_e'(w), w) - (\gamma_c' - 1)z_c'(w) - (\gamma_e' - 1)z_e'(w))\right] dF(w) = -\int \left[\Lambda_c(w) + \Lambda_e(w)\right] dF(w) = 0.$$

For the first equality, the construction of a new policy and a new equilibrium allocation was used. The second one can be simply derived after some algebra and using the fact that the corresponding constraint in the original equilibrium must hold. □

Proof of Proposition 3. First, assume that the condition holds.
Consider the policy $\{\gamma_c, \gamma_e, t_c, \{t_e(z, w)\}_w\}$, the equilibrium allocation $\{(q(w), \hat{z}_c(w), \hat{z}_e(w))\}_w$, and CBDC transfer function $t_e(z, w)$:

$$t_e(z, w) = \begin{cases} \frac{(\gamma_e - 1)D_w(q_w^*)}{\gamma_e} & z \geq \frac{D_w(q_w^*)}{\gamma_e} \\ 0 & z < z_e(w) \end{cases}.$$

$t_c = 0$, $\gamma_e > 1$, and $\gamma_c > 1$ is set sufficiently high that $\max_q\{-(\gamma-\beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\} = 0$. Under this policy, no buyer wants to carry any cash, as their payoff would be 0. Moreover, the buyer would take exactly $z_e(w) = \frac{D_w(q_w^*)}{\gamma_e}$ real balances from the CM. The constraint of the planner's problem is satisfied by the construction of t_e.

Second, assume that the first best is achievable. By Proposition 2, cash is not used without loss of generality, so we can assume that $t_c = 0$. Following the second constraint of the planner's problem, one obtains $t_e(z,w) = (\gamma_e - 1)z_e(w)$. Since first best is achievable, the balances that the agent takes from the CM plus the transfers that the agent receives, $z_e(w) + t_e(z_e(w), w)$, must be equal to $D_w(q_w^*)$. Therefore, $z_e(w) = \frac{D_w(q_w^*)}{\gamma_e}$, and consequently, $t_e(z,w) = (\gamma_e - 1)z_e(w) = \frac{(\gamma_e-1)D_w(q_w^*)}{\gamma_e}$. The buyer must get a positive payoff, so

$$-(\gamma_e - \beta)D_w(q_w^*) + \beta\sigma\theta(wu(q_w^*) - c(q_w^*)) + \gamma_e t_e(z_e(w), w) \geq 0.$$

Otherwise, the buyer would skip the CM and DM; i.e., there would be no trade. Simple algebra ensures that the condition must hold using the values derived for $z_e(w)$ and $t_e(z_e(w), w)$. □

Proof of Remark 1. Define $r(q)$ as follows:

$$r(q) \equiv \frac{c'(q)}{u'(q)} \Rightarrow r' = \frac{c'}{u'}\left(\frac{c''}{c'} - \frac{u''}{u'}\right) \text{ and } q_w^* = r^{-1}(w).$$

The arguments of the functions are eliminated when there is no danger of confusion. Therefore,

$$\bar{\theta}(w) \text{ is decreasing } \Leftrightarrow \frac{wu(q_w^*)}{c(q_w^*)} \text{ is increasing } \Leftrightarrow \frac{\partial}{\partial w}\left(\frac{wu(r^{-1}(w)}{c(r^{-1}(w))}\right) > 0.$$

Now,

$$c^2\frac{\partial}{\partial w}\left(\frac{wu(r^{-1}(w)}{c(r^{-1}(w))}\right) = (u + wu'/r')c - wuc'/r'$$
$$= uc\left(1 + w/r'(u'/u - c'/c)\right) = uc\left(1 + \frac{u'/u - c'/c}{c''/c' - u''/u'}\right),$$

where in the last step, the fact that $wu'(q_w^*) = c'(q_w^*)$ was used. But

$$1 + \frac{u'/u - c'/c}{c''/c' - u''/u'} > 0 \Leftrightarrow \frac{cc'' - c'^2}{cc'} > \frac{uu'' - u'^2}{uu'} \Leftrightarrow \frac{\partial}{\partial q} ln\left(\frac{c'(q)u(q)}{c(q)u'(q)}\right) > 0.$$

□

Proof of Proposition 4. Both means are costless, so we can assume without loss of generality that inflation is sufficiently high so that cash is not used and also t_e is characterized as a step function. Since the payoff from using cash is 0, the following must hold at any equilibrium:

$$-(\gamma_e - \beta)D_w(q(w)) + \beta\sigma\theta\Big(wu(q(w)) - c(q(w))\Big) + \gamma_e t_e(z(w), w) \geq 0 \text{ for all } w,$$

$$\int_w (t_e(z(w), w) - (\gamma_e - 1)z(w))dF(w) \leq 0.$$

One can write from the first constraint that $-\gamma_e t_e(z(w), w) = -(\gamma_e - \beta)D_w(q(w)) + \beta\sigma\theta(wu(q(w)) - c(q(w))) - \epsilon(w)$ for some $\epsilon(w) \geq 0$. Since $t_e(z(w), w) + z(w) = D_w(q(w))$, the budget constraint can be written as

$$0 = 1/\gamma_e \Bigg(\int (\gamma_e t_e(z(w), w) - \gamma(\gamma_e - 1)z(w))dF(w) \Bigg)$$

$$= 1/\gamma_e \Bigg((\gamma_e - \beta)D_w(q(w)) - \beta\sigma\theta(wu(q(w)) - c(q(w))) + \epsilon(w)$$

$$-(\gamma_e - 1)(\beta D_w(q(w)) + \beta\sigma\theta(wu(q(w)) - c(q(w)) - \epsilon(w)) \Bigg) dF(w)$$

$$= \int ((1 - \beta)D_w(q(w)) + \beta\sigma\theta(wu(q(w)) - c(q(w))) + \epsilon(w))dF(w). \tag{23}$$

"only if" part: Now suppose the first best is achievable. Since $\epsilon(w) \geq 0$ for all w, the condition in the proposition will follow.

"if" part: If the condition is satisfied, then it is easy to verify that the first best is achievable with the following choice of $t_e(z, w)$:

$$t_e(z, w) = \begin{cases} \frac{(\gamma_e - \beta)D_w(q^*(w)) - \beta\sigma\theta(wu(q^*(w)) - c(q^*(w))) + v}{\gamma_e} & z \geq \frac{\beta D_w(q^*(w)) + \beta\sigma\theta(wu(q^*(w)) - c(q^*(w))) - v}{\gamma_e} \\ 0 & \text{otherwise} \end{cases},$$

where

$$v \equiv \int -(1 - \beta)D_w(q^*(w)) + \beta\sigma\theta(wu(q^*(w)) - c(q^*(w)))dF(w).$$

Also, set $t_c = 0$, and again $\gamma_c > 1$ is set sufficiently high that $\max_q\{-(\gamma_c - \beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\} = 0$, and γ_e is set sufficiently high that $-(\gamma_e - \beta)D_w(q^*(w)) + \beta\sigma\theta(wu(q^*(w)) - c(q^*(w))) \leq 0$. Note that $v \geq 0$ following the assumption in the proposition. Notice that I have assumed the distribution is such that $v \leq \frac{\beta D_w(q^*(w)) + \beta\sigma\theta(wu(q^*(w)) - c(q^*(w)))}{\gamma_e}$ for all types. If it does not hold for some types, then those types do not need to bring any balances. In that case, the transfer scheme needs to be modified easily, but I skip it for the sake of brevity. □

Proof of Proposition 6.

$$K_2(w) \equiv s(w, q_w^*) \geq K_1(w) \equiv \beta\sigma f(w, q_w^*) - \beta\sigma f(w, \bar{q}(w, 1))$$

$$\Leftrightarrow \beta\sigma f(w, \bar{q}(w, 1)) \geq \beta\sigma f(w, q_w^*) - s(w, q_w^*)$$

$$\beta\sigma(wu(\bar{q}(w, 1)) - c(\bar{q}(w, 1))) \geq (1 - \beta)D_w(q_w^*) + \beta\sigma(1 - \theta)(wu(q_w^*) - c(q_w^*))$$

The RHS is decreasing and the LHS is increasing in θ. As $\theta \to 1$, the LHS is equal to $\{\max_q -(1-\beta)c(q) + \beta\sigma(wu(q) - c(q))\} + (1-\beta)c(\bar{q}(w,1))$, which is greater than the RHS, $(1-\beta)c(q_w^*)$, for β sufficiently close to 1. This concludes the proof. □

Proof of Proposition 7. I prove this result in three steps. In Lemma 2, I show that a necessary condition for co-existence is that agents using CBDC must be indifferent between cash and CBDC. In Lemma 3, I rewrite the planner's problem in a simpler form. Finally, I write the necessary conditions for optimality and prove the result.

Lemma 2. *Suppose co-existence is optimal; i.e., it is optimal that both cash and CBDC are used. Then it can be assumed without loss of generality that all types who use CBDC are indifferent between using cash and CBDC.*

Proof of Lemma 2. Suppose the constraint is slack in Problem 2 for a strictly positive measure of types, so they strictly prefer to use CBDC. Then, the planner requires these types to bring more balances but the planner keeps $z_e + t_e$ constant for each type. The planner then uses the remaining balances to transfer to other types who use CBDC but do not get to the first best. If all types who use CBDC get their first-best level of consumption, the planner will distribute the remaining balances in the form of cash across everybody. This policy does not reduce welfare, as it does not change the incentives of buyers to bring enough balances. Therefore, we can assume without loss of generality that CBDC users are indifferent between cash and CBDC. □

Lemma 3. *Given γ_c, the planner's problem can be written as follows:*

$$\max_{\tilde{q}(w,\gamma_c), e(w,\gamma_c)} \int \Big[(\beta\sigma f(w,\tilde{q}(w,\gamma_c)) - K)e(w,\gamma_c) + \beta\sigma f(w,\bar{q}(w,\gamma_c))(1-e(w,\gamma_c))\Big]dF$$

$$s.t. \int \Big(s(w,\tilde{q}(w,\gamma_c)) - K - s(w,\bar{q}(w,\gamma_c))\Big)e(w,\gamma_c)dF + \int (\gamma_c - 1)D_w(\bar{q}(w,\gamma_c))dF \geq 0,$$

where $\tilde{q}(w,\gamma_c)$ is the buyer of type w's consumption in the DM when the buyer uses CBDC and the cash inflation rate is γ_c.

Proof of Lemma 3. If a type w buyer wants to use CBDC, his payoff from using CBDC must be higher than that from cash. Therefore, for any type who uses CBDC under the inflation rates γ_c and γ_e, there should exist $\epsilon(w) \geq 0$ such that:

$$\underbrace{-(\gamma_e - \beta)(\hat{z}_e + t_e(\hat{z}_e, w)) + \beta\sigma\theta(wu(\tilde{q}) - c(\tilde{q}))) - K + \gamma_e t_e(\hat{z}_e, w) + \beta t_c}_{\text{gains from using CBDC}}$$

$$= \underbrace{-(\gamma_c - \beta)(\hat{z}_c + t_c) + \beta\sigma\theta(wu(\bar{q}) - c(\bar{q})) + \gamma_c t_c}_{\text{gains from using cash}} + \epsilon(w),$$

where $D_w(\bar{q}) = \hat{z}_c + t_c$ and $D_w(\tilde{q}) = \hat{z}_e + t_e(\hat{z}_e) + t_c$.[32] After some algebra, this constraint reduces to:

$$s(w, \tilde{q}) - K - (\gamma_e - 1)\hat{z}_e + t_e(\hat{z}_e) + (1 - \gamma_c)t_c = s(w, \bar{q}) - (\gamma_c - 1)(\hat{z}_c + t_c) + \epsilon(w).$$

Now, consider the constraint of problem 2. Using the fact that each type carries either cash or CBDC, one can write:

$$t_c + \int \Big[\big(t_e(z_e(w), w) - (\gamma_e - 1)z_e(w)\big)e(w, \gamma) - (\gamma_c - 1)z_c(w)(1 - e(w, \gamma))\Big] dF(w) = 0.$$

Combining the last two equations, one yields

$$\int (s(w, \tilde{q}(w, \gamma_c)) - K - s(w, \bar{q}(w, \gamma_c)))e(w, \gamma_c)dF + \int (\gamma_c - 1)D_w(\bar{q}(w, \gamma))dF$$

$$= \gamma_c t_c + \int \epsilon(w)e(w, \gamma)dF.$$

Note that $\epsilon(w) \geq 0$, and that t_c does not show up anywhere in the objective function and in the LHS of this constraint; therefore, we can simply replace this constraint with another one where the LHS is greater than or equal to 0, as $t_c \geq 0$. □

Following Lemma 3, the Lagrangian for the planner's problem can be written as follows:

$$\mathcal{L} = \int \Big[\big(\beta\sigma f(w, \tilde{q}(w, \gamma_c)) - K - \beta\sigma f(w, \bar{q}(w, \gamma_c))\big)e(w, \gamma) + \beta\sigma f(w, \bar{q}(w, \gamma_c))$$

$$+\lambda(\gamma_c)\big[\big(s(w, \tilde{q}(w, \gamma_c)) - K - s(w, \bar{q}(w, \gamma_c))\big)e(w, \gamma_c) + (\gamma_c - 1)D_w(\bar{q}(w, \gamma_c))\big]$$

$$+\mu_1(w)e(w, \gamma_c) + \mu_2(w)(1 - e(w, \gamma_c))\Big] dF,$$

where the Lagrangian multipliers are denoted by λ, $\lambda\mu_1$ and $\lambda\mu_2$. Note that CBDC inflation is irrelevant as long as it is sufficiently high. Necessary conditions for optimality imply that $\tilde{q}(w, \gamma)$ solves

$$\beta\sigma f_q(w, q) + \lambda(\gamma)s_q(w, q) = 0.$$

[32] The payoff from using CBDC has been written given the fact that if type w wants to use CBDC, he will use only CBDC. This is true due to the following: First, the cost of carrying CBDC is independent of how many balances the buyer carries. Second, the transfers in CBDC are in the form of a step function, so if a buyer already incurs the cost of CBDC, it is not worth it to have some balances in cash.

As a result,

$$\tilde{q}(w,\gamma) \in [\bar{q}(w,\gamma), q_w^*].$$

Other necessary conditions for optimality:

$$e(w,\gamma)=1 \rightarrow \beta\sigma f(w,\tilde{q}(w,\gamma))-K-\beta\sigma f(w,\bar{q}(w,\gamma))+\lambda(\gamma)\Big[s(w,\tilde{q}(w,\gamma))-K-s(w,\bar{q}(w,\gamma))\Big]=\mu_2 \geq 0,$$

$$e(w,\gamma)=0 \rightarrow \beta\sigma f(w,\tilde{q}(w,\gamma))-K-\beta\sigma f(w,\bar{q}(w,\gamma))+\lambda(\gamma)\Big[s(w,\tilde{q}(w,\gamma))-K-s(w,\bar{q}(w,\gamma))\Big]=-\mu_1 \leq 0.$$

Now, by way of contradiction, assume that $e(w_1,\gamma_c)=1$, $e(w_2,\gamma_c)=0$, for $w_1<w_2$. I show below that the following is increasing in w:

$$\beta\sigma f(w,\tilde{q}(w,\gamma))-K-\beta\sigma f(w,\bar{q}(w,\gamma))+\lambda(\gamma)\big(s(w,\tilde{q}(w,\gamma))-K-s(w,\bar{q}(w,\gamma))\big).$$

But this is in contradiction with the necessary conditions for optimality. The above term is equal to:

$$\int_{\bar{q}(w,\gamma)}^{\tilde{q}(w,\gamma)} \big[\beta\sigma f_q(w,q)+\lambda(\gamma)s_q(w,q)\big]dq-K(1+\lambda(\gamma)).$$

As shown above, $\bar{q}(w,\gamma_c)\leq\tilde{q}(w,\gamma_c)$ for all w for which $e(w,\gamma_c)=1$. One needs to show that the derivative of the above expression with respect to w is positive. Therefore, it suffices to show

$$\int_{\bar{q}(w,\gamma)}^{\tilde{q}(w,\gamma)} \big[\underbrace{\beta\sigma f_{wq}(w,q)}_{\geq 0}+\lambda(\gamma_c)\underbrace{s_{wq}(w,q)}_{\geq 0}\big]dq$$
$$+\frac{\partial\tilde{q}(w,\gamma_c)}{\partial w}\int_{q_0}^{\tilde{q}(w,\gamma_c)}\big[\beta\sigma f_q(w,q)+\lambda(\gamma_c)s_q(w,q)\big]dq$$
$$-\frac{\partial\bar{q}(w,\gamma_c)}{\partial w}\int_{q_0}^{\bar{q}(w,\gamma_c)}\big[\beta\sigma f_q(w,q)+\lambda(\gamma_c)s_q(w,q)\big]dq\geq 0.$$

Define $\tilde{\alpha}$ and $\bar{\alpha}$ as follows:

$$\tilde{q}(w,\gamma)=Q(\tilde{\alpha}w) \text{ where } \tilde{\alpha}\equiv\frac{-(1-\beta)(1-\theta)+\beta\sigma(\theta+1/\lambda(\gamma))}{(1-\beta)\theta+\beta\sigma(\theta+1/\lambda(\gamma))},$$

$$\bar{q}(w,\gamma)=Q(\bar{\alpha}w) \text{ where } \bar{\alpha}\equiv\frac{-(\gamma-\beta)(1-\theta)+\beta\sigma\theta}{(\gamma-\beta)\theta+\beta\sigma\theta}.$$

But $\bar{q}(w,\gamma)\leq\tilde{q}(w,\gamma)$, so $\tilde{\alpha}\geq\bar{\alpha}$. It is shown in Lemma 4 below that $\alpha Q'(\alpha w)$ is increasing in α, so $\tilde{\alpha}Q'(\tilde{\alpha}w)\geq\bar{\alpha}Q'(\bar{\alpha}w)$. Therefore,

$$\Rightarrow\frac{\partial\tilde{q}(w,\gamma)}{\partial w}=\frac{\partial Q(\tilde{\alpha}w)}{\partial w}=\tilde{\alpha}Q'(\tilde{\alpha}w)\geq\bar{\alpha}Q'(\bar{\alpha}w)=\frac{\partial Q(\bar{\alpha}w)}{\partial w}=\frac{\partial\bar{q}(w,\gamma)}{\partial w}.$$

Finally, $\tilde{q}(w,\gamma)\geq\bar{q}(w,\gamma)$, $Q'(.)$ is positive, and $f_q(w,q)+\lambda(\gamma)s_q(w,q)>0$ for $q<\tilde{q}(w,\gamma)$, so the inequality is established. □

Lemma 4. *If* $\frac{wQ''(w)}{Q'(w)} \geq -1$*, then* $\alpha Q'(\alpha w)$ *is increasing in* α*.*

Proof.

$$\frac{\partial}{\partial \alpha}\Big(\alpha Q'(\alpha w)\Big) \geq 0 \Leftrightarrow Q'(\alpha w) + \alpha w Q''(\alpha w) \geq 0 \Leftrightarrow -\frac{\alpha w Q''(\alpha w)}{Q'(\alpha w)} \leq 1.$$

□

Derivation of the constraint of the planner's problem in the two-type example

The first constraint is simplified to:

$$t_{e2} = (\gamma_e - 1)z_{e2} + (1-\pi)/\pi(\gamma_c - 1)D_{w_1}(q_1) - ((1-\pi)\gamma_c + \pi)t_c/\pi,$$

where z_{c1} is substituted out from the third constraint. Using the other constraints, one yields:

$$t_c + z_{e2} + (\gamma_e - 1)z_{e2} + (1-\pi)/\pi(\gamma_c - 1)D_{w_1}(q_1) - ((1-\pi)\gamma_c + \pi)t_c/\pi = D_{w_2}(q_2)$$

$$\Rightarrow \gamma_e z_{e2} = D_{w_2}(q_2) - (1-\pi)/\pi(\gamma_c - 1)D_{w_1}(q_1) + (1-\pi)\gamma_c t_c/\pi$$

$$\Rightarrow \gamma_e t_{e2} = (\gamma_e - 1)D_{w_2}(q_2) + (1-\pi)/\pi(\gamma_c - 1)D_{w_1}(q_1) - (1-\pi)\gamma_c t_c/\pi - \gamma_e t_c.$$

Now we can insert $\gamma_e t_{e2}$ back in the Incentive Constraint to derive (13).

Proof of Proposition 8. Assume that $\gamma_c \leq 1$. Then,

$$\max_q\{-(\gamma_c - \beta)D_{w_2}(q) + \beta\sigma\theta(w_2 u(q) - c(q))\}$$

$$\geq \max_q\{-(1-\beta)D_{w_2}(q) + \beta\sigma\theta(w_2 u(q) - c(q))\}$$

$$> -(1-\beta)D_{w_2}(q_2) + \beta\sigma\theta(w_2 u(q_2) - c(q_2)) - K + (1-\pi)/\pi(\gamma_c - 1)D_{w_1}(q_1) - \gamma_c t_c/\pi,$$

where the first inequality comes from $\gamma_c \leq 1$, and the second inequality comes from $K > 0$ and $t_c \geq 0$. This is a contradiction, because under the co-existence scheme, (13) should hold.

□

Proof of Proposition 9. $\bar{B}(\pi)$ is a continuous function of π around $\pi = 1$, using the theorem of the maximum. $E(\pi)$ is also continuous. By Assumption 2, I show below that $\bar{B}(1) < E(1)$. Therefore, there exists a neighborhood around $\pi = 1$ such that $\bar{B}(\pi) < E(\pi)$ for $\pi > \tilde{\pi}$ where $\tilde{\pi} \in (0, 1)$. This completes the proof.

Proof of $\bar{B}(1) < E(1)$: To show this, we need to show that $q_2 = q_2^*$ is not feasible for the maximization problem $\bar{B}$ at $\pi = 1$. That is, we need to show that as $\pi \to 1$, (13) does not

hold for any possible γ when $q_2 = q_2^*$.
Note that $\gamma_0 < \gamma_1$, iff $O(w_2, \gamma_0) > O(w_2, \gamma_1)$. Therefore, for a given $\gamma < \gamma_0$, we have

$$\begin{aligned} O(w_2, \gamma) - (1-\pi)/\pi(\gamma - 1)D_{w_1}(\bar{q}_1(\gamma)) &> O(w_2, \gamma_0) > O(w_2, \gamma_1) \\ &= -(1-\beta)D_{w_2}(q_2^*) + \beta\sigma\theta(w_2 u(q_2^*) - c(q_2^*)) - K. \end{aligned}$$

The first inequality is due to the fact that $(\gamma - 1)D_{w_1}(q_1)$ is bounded for $\gamma < \gamma_0$ and π is sufficiently close to 1, and the equality holds by definition of γ_1. Notice that we do not need to consider any $\gamma \geq \gamma_0$, because such allocation would be dominated for type 1. □

Proof of Proposition 10; Privacy in the two-type example. As mentioned earlier, under the cash-only scheme, welfare with private information is unchanged compared with complete information, because there is no communication in the economy with only cash and the only policy feasible toward cash is helicopter drop. Here, I consider 3 cases and calculate welfare with private information. In case 1, I consider CBDC-only scheme, in case 2, the co-existence scheme, and in case 3, the co-existence-2 scheme in which type 1 uses CBDC and type 2 uses cash. For the CBDC-only scheme, I derive conditions under which welfare with private information is less than that with complete information. For the co-existence scheme, I show that the IC_{12} is not binding as long as θ is close to 1. For the co-existence-2 scheme, it is shown that welfare is less than that under the co-existence scheme, therefore, one can simply ignore the co-existence-2 scheme and compare the same three schemes that we considered in the complete information case: CBDC-only, cash-only and co-existence.

To ease notation, I often demonstrate X_i as a short form for X_{w_i}, e.g., $D_1(q_1)$ is a short form for $D_{w_1}(q_{w_1})$.

Case 1. CBDC-only scheme

I first suppose that the first best is achievable for both types if the planner has complete information. I then find conditions under which this allocation is not incentive compatible. Hence, under such conditions, the welfare with private information is strictly less than that with complete information.

First, consider the **complete information case**. There are no IC constraints, so the first best is achievable iff t_1 and t_2 can be found to satisfy the following:

$$\begin{aligned} (1-\pi)t_1 + \pi t_2 &= \frac{\gamma_e - 1}{\gamma_e}((1-\pi)D_1(q_1^*) + \pi D_2(q_2^*)), \\ IR_1 &: -(\gamma_e - \beta)D_1(q_1^*) + \beta\sigma\theta f_1(q_1^*) - K + \gamma_e t_1 \geq 0, \\ IR_2 &: -(\gamma_e - \beta)D_2(q_2^*) + \beta\sigma\theta f_2(q_2^*) - K + \gamma_e t_2 \geq 0, \end{aligned}$$

which is equivalent to:

$$\beta\sigma\theta[(1-\pi)f_1(q_1^*)+\pi f_2(q_2^*)]-K \geq (1-\beta)\left[(1-\pi)D_1(q_1^*)+\pi D_1(q_2^*)\right].$$

This is identical to the condition in Proposition 4 for the two-type case and when $K=0$.

Now, consider the **private information case**. The planner's problem is as follows:

$$\max_{\{\gamma_e,t_c,\{t_e(z_e,w_i)\}\}} \sum_{i=1,2} \pi_i(\beta\sigma f_i(q_i)-K)$$

subject to:

$$\begin{aligned}
\text{payment} &: D_i(q_{ij}) = min\{D_i(q_i^*), t_j+z_j\},\\
\text{Planner's budget} &: (1-\pi)t_1+\pi t_2 = (\gamma_e-1)((1-\pi)z_1+\pi z_2)\\
\text{IC}_{ij}\ (\text{type } i \text{ doesn't report } j) &: \Pi_{ii} \geq \Pi_{ij},\\
\text{IR}_i\ (\text{type i doesn't want cash}) &: \Pi_{ii} \geq 0\ \forall i,
\end{aligned}$$

where

$$\Pi_{ij} \equiv -(\gamma_e-\beta)(t_j+z_j)+\beta\sigma\theta f_i(q_{ij})-K+\gamma_e t_j.$$

The first constraint states that if type i reports type j, then type i can make a payment up to t_j+z_j, which is the purchasing power of type j. If type 1 reports type 2, then type i will have more resources than they could buy at the first best level, therefore:

$$\begin{aligned}
q_{12} &= q_1^*\\
q_{21} &= D_2^{-1}(D_1(q_1^*))
\end{aligned}$$

Therefore, the IC constraints can be rewritten as:

$$\begin{aligned}
IC_{12} &: -(\gamma_e-\beta)(t_1+z_1)+\beta\sigma\theta f_1(q_1^*)+\gamma_e t_1 \geq -(\gamma_e-\beta)(t_2+z_2)+\beta\sigma\theta f_1(q_1^*)+\gamma_e t_2,\\
IC_{21} &: -(\gamma_e-\beta)(t_2+z_2)+\beta\sigma\theta f_2(q_2^*)+\gamma_e t_2 \geq -(\gamma_e-\beta)(t_1+z_1)+\beta\sigma\theta f_2(D_2^{-1}(D_1(q_1^*)))+\gamma_e t_1.
\end{aligned}$$

They, together, are simplified to:

$$\begin{aligned}
& f_2(q_2^*)-f_2(D_2^{-1}(D_1(q_1^*))) \geq \frac{-(\gamma_e-\beta)(t_1+z_1-t_2-z_2)+\gamma_e(t_1-t_2)}{\beta\sigma\theta} \geq 0,\\
\Leftrightarrow\ & f_2(q_2^*)-f_2(D_2^{-1}(D_1(q_1^*))) \geq \frac{(\gamma_e-\beta)(D_2(q_2^*)-D_1(q_1^*))+\gamma_e(t_1-t_2)}{\beta\sigma\theta} \geq 0, \qquad (24)
\end{aligned}$$

where I used $t_1+z_1=D_1(q_1^*)$ and $t_2+z_2=D_2(q_2^*)$.

I characterize the conditions under which the first best allocation is not supported. Denote by (t_1^a,t_2^a) the solution for the RHS inequality in (24), and denote by (t_1^a,t_2^a) the solution

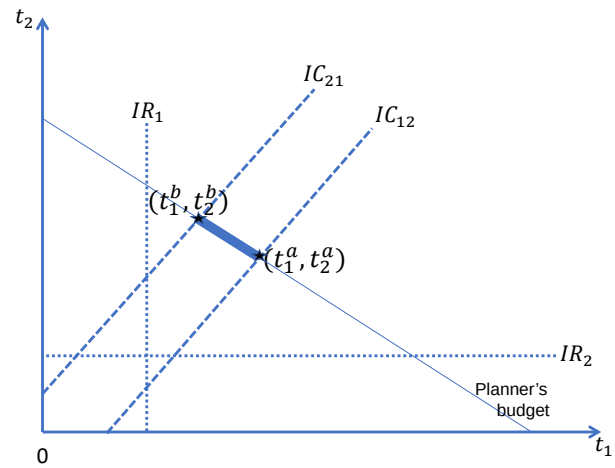

Figure 11: Characterization of planner's problem with private information

for the LHS inequality in (24), when they hold with equality and also satisfy the planner's budget constraint. See Figure 11 for illustration.

$$\begin{aligned}
\gamma_e(t_2^a - t_1^a) &= (\gamma_e - \beta)\left(D_2(q_2^*) - D_1(q_1^*)\right) \\
\gamma_e(t_2^b - t_1^b) &= -\beta\sigma\theta\left(f_2(q_2^*) - f_2(D_2^{-1}(D_1(q_1^*)))\right) + (\gamma_e - \beta)\left(D_2(q_2^*) - D_1(q_1^*)\right) \\
(1-\pi)t_1^a + \pi t_2^a &= (1-\pi)t_1^b + \pi t_2^b = \frac{\gamma_e - 1}{\gamma_e}((1-\pi)D_1(q_1^*) + \pi D_2(q_2^*))
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \gamma_e t_1^a &= \begin{pmatrix} \gamma_e - \beta \\ -(1-\pi)(1-\beta) \end{pmatrix} D_1(q_1^*) - \pi(1-\beta)D_2(q_2^*) \\
\Rightarrow \quad \gamma_e t_2^b &= \begin{pmatrix} \gamma_e - \beta \\ -\pi(1-\beta) \end{pmatrix} D_2(q_2^*) - (1-\pi)(1-\beta)D_1(q_1^*) \\
&\quad -(1-\pi)\beta\sigma\theta\left(f_2(q_2^*) - f_2(D_2^{-1}(D_1(q_1^*)))\right).
\end{aligned}$$

Note that t_1^a gives us the highest t_1 such that the IC is satisfied, so if IR_1 is not satisfied, then we cannot achieve the first best. Similarly, t_2^b gives us the highest t_2 such that the IC is satisfied, so if IR_2 is not satisfied, then we cannot achieve the first best. We want IR_1 or IR_2 violated. For IR_1 to be violated, we must have:

$$\begin{aligned}
&-(\gamma_e - \beta)D_1(q_1^*) + \beta\sigma\theta f_1(q_1^*) + \begin{pmatrix} \gamma_e - \beta \\ -(1-\pi)(1-\beta) \end{pmatrix} D_1(q_1^*) - \pi(1-\beta)D_2(q_2^*) - K < 0 \\
\Rightarrow \quad &\beta\sigma\theta f_1(q_1^*) - K < (1-\beta)\left[(1-\pi)D_1(q_1^*) + \pi D_2(q_2^*)\right]
\end{aligned}$$

For IR_2 to be violated, we must have:

$$\left[\begin{array}{c} -(\gamma_e-\beta)D_2(q_2^*)+\beta\sigma\theta f_2(q_2^*) \\ +\begin{pmatrix}\gamma_e-\beta \\ -\pi(1-\beta)\end{pmatrix}D_2(q_2^*)-(1-\pi)(1-\beta)D_1(q_1^*) \\ -(1-\pi)\beta\sigma\theta\left(f_2(q_2^*)-f_2(D_2^{-1}(D_1(q_1^*)))\right)-K \end{array}\right]<0,$$

$$\Rightarrow \quad \beta\sigma\theta\left[\pi f_2(q_2^*)+(1-\pi)f_2(D_2^{-1}(D_1(q_1^*)))\right]-K<(1-\beta)\left[(1-\pi)D_1(q_1^*)+\pi D_2(q_2^*)\right].$$

In summary, if

$$\begin{aligned}\min\left\{\left[\begin{array}{c}(1-\pi)f_2(D_2^{-1}(D_1(q_1^*)))\\ +\pi f_2(q_2^*)\end{array}\right],f_1(q_1^*)\right\} &< \frac{1-\beta}{\beta\sigma\theta}\left[(1-\pi)D_1(q_1^*)+\pi D_1(q_2^*)\right]+\frac{K}{\beta\sigma\theta}\\ &\leq (1-\pi)f_1(q_1^*)+\pi f_2(q_2^*),\end{aligned}$$

then the first best is achievable with complete information but not with private information. This is the same condition as (14).

Case 2. Co-Existence Scheme

I show that if co-existence is optimal, the IC constraints are not binding. Of course, type 2 does not want to report type 1, in which case he should use cash but this constraint has been already taken into account.

The planner's problem can be written as follows:

$$\overline{B} \equiv \max_{t_c,t_2,z_{c1},z_{e2},\gamma_c,\gamma_e,q_2} B$$

subject to:

$$\begin{aligned} t_c+\pi_2(t_2-(\gamma_e-1)z_{e2}) &= (1-\pi_2)(\gamma_c-1)z_{c1},\\ D_2(q_2) &= \min\{D_2(q_2^*),t_c+t_2+z_{e2}\},\\ D_1(q_1) &= \min\{D_1(q_1^*),t_c+z_{c1}\},\end{aligned}$$

$$\begin{aligned}\text{IC}_{21} &: \ O(w_2,\gamma_c)\leq -(\gamma_e-\beta)(t_2+z_{e2})-(\gamma_c-\beta)t_c+\beta\sigma\theta(w_2u(q_2)-c(q_2))-K+\gamma_e t_2,\\ \text{IC}_{12} &: \ O(w_1,\gamma_c)\geq -(\gamma_e-\beta)(t_2+z_{e2})-(\gamma_c-\beta)t_c+\beta\sigma\theta(w_1u(q_{12})-c(q_{12}))-K+\gamma_e t_{e2},\end{aligned}$$

where

$$D_1(q_{12})=min\{D_1(q_1^*),t_2+z_2\}.$$

IC_{21} can be simplified to

$$O(w_2,\gamma_c)\leq -(1-\beta)D_{w_2}(q_2)+\beta\sigma\theta(w_2u(q_2)-c(q_2))-K+(1-\pi_2)/\pi_2(\gamma_c-1)D_{w_1}(\bar{q}_1(\gamma_c)),$$

which is the same as (13). IC_{12} can be simplified to

$$\begin{aligned}
& -O(w_1,\gamma_c)+O(w_2,\gamma_c) \\
& -(\gamma_e-\beta)(t_{e2}+z_{e2})-(\gamma_c-\beta)t_c+\beta\sigma\theta(w_2u(q_2)-c(q_2))-K+\gamma_e t_{e2}-O(w_2,\gamma_c) \\
\leq\ & \beta\sigma\theta\left[f(w_2,D_2^{-1}(t_2+z_2))-f(w_1,D_1^{-1}(min\{D_1(q_1^*),t_2+z_2\}))\right].
\end{aligned}$$

I show that IC_{12} is not binding at the solution. To do that, I consider two cases: **a** and **b**. Remember from the complete information case that at the solution, $q<q_2^*$.

Case a: IC_{21} is non-binding.
I rewrite IC_{12} as

$$\begin{aligned}
& -O(w_1,\gamma_c)+O(w_2,\gamma_c) \\
& -(1-\beta)D_{w_2}(q_2)+\beta\sigma\theta(w_2u(q_2)-c(q_2))-K+(1-\pi_2)/\pi_2(\gamma_c-1)D_{w_1}(q_1) \\
\leq\ & \beta\sigma\theta\left[f(w_2,D_2^{-1}(t_2+z_2))-f(w_1,D_1^{-1}(min\{D_1(q_1^*),t_2+z_2\}))\right].
\end{aligned}$$

If IC_{21} is non-binding, we can increase $t_{e2}+z_{e2}$ by a small amount and the objective function increases. The RHS of this constraint also increases according to the following Claim.[33] Its LHS decreases, therefore IC_{12} is not violated. IC_{21} is also not violated either, because it was non-binding and the change in $t_{e2}+z_{e2}$ is sufficiently small. This is a contradiction, therefore, IC_{21} should be binding.

Claim: $f(w_2,D_2^{-1}(z))-f(w_1,D_1^{-1}(z))$ is increasing in z for $z\leq D_2(q_{w2}^*)$.

Proof of the Claim:

$$d\frac{f(w_2,D_{w_2}^{-1}(z))-f(w_1,D_{w_1}^{-1}(z))}{dz}=\frac{f_q(w_2,D_2^{-1}(z))}{D'_{w_2}(D_{w_2}^{-1}(z))}-\frac{f_q(w_1,D_{w_1}^{-1}(z))}{D'_{w_1}(D_{w_1}^{-1}(z))}$$

It suffices to show that $\frac{f_q(w,D_w^{-1}(z))}{D'_w(D_w^{-1}(z))}$ is increasing in w. Note that:

$$\frac{f_q(w,q)}{D'_w(q)}=\frac{wu'(q)-c'(q)}{(1-\theta)wu'(q)+\theta c'(q)}=\frac{1}{1-\theta}\left(1-\frac{1}{(1-\theta)w\frac{u'(q)}{c'(q)}+\theta}\right)$$

where $D_w(q)=z$. Since $\frac{u'(q)}{c'(q)}$ is decreasing in q and q is decreasing in w, the denominator is increasing in w, so is the whole expression. This completes the proof.

Case b: IC_{21} is binding.

[33] Note that if $D_1(q_1^*)\leq t_2+z_2$, the statement is clearly true because $f(w_2,D_2^{-1}(t_2+z_2))$ is increasing in t_2+z_2.

In this case, IC_{12} is equivalent to:

$$-O(w_1,\gamma_c)+O(w_2,\gamma_c)\leq\beta\sigma\theta\left[f(w_2,D_2^{-1}(t_2+z_2))-f(w_1,min\{q_1^*,D_1^{-1}(t_2+z_2)\})\right].$$

Since $min\{q_1^*,D_1^{-1}(t_2+z_2)\})\leq D_1^{-1}(t_2+z_2)$, it is sufficient to show that the following inequality cannot be binding:

$$-O(w_1,\gamma_c)+O(w_2,\gamma_c)\leq\beta\sigma\theta\left[f(w_2,D_2^{-1}(t_2+z_2))-f(w_1,D_1^{-1}(t_2+z_2))\right].$$

I show below that the latter holds true for $\theta=1$. At $\theta=1$, $D_1(q)=D_2(q)=c(q)$, therefore, $D_1^{-1}(t_2+z_2)=D_2^{-1}(t_2+z_2)=c^{-1}(t_2+z_2)=q_2$. Hence, I need to show

$$\begin{aligned}
& -O(w_1,\gamma_c)+O(w_2,\gamma_c)<\beta\sigma\theta\left[f(w_2,q_2)-f(w_1,q_2)\right],\\
\Leftrightarrow\quad & -O(w_1,\gamma_c)+O(w_2,\gamma_c)<\beta\sigma\theta(w_2-w_1)u(q_2),\\
\Leftrightarrow\quad & \int_{w_1}^{w_2}\beta\sigma\theta u(\bar{q}_w(\gamma_c))dw<\int_{w_1}^{w_2}\beta\sigma\theta u(q_2)dw.
\end{aligned}$$

Hence, it is sufficient to show

$$u(\bar{q}_w(\gamma_c))\leq u(q_2)\text{ for all }w\in(w_1,w_2),$$

with inequality for a subset of them. Given that $\bar{q}_w(\gamma_c)$ is strictly increasing in w, it is enough to show that $u(\bar{q}_{w_2}(\gamma_c))\leq u(q_2)$, or equivalently, $\bar{q}_{w_2}(\gamma_c)\leq q_2$. This is correct given that q_2 is part of the solution to the planner's maximization under the co-existence scheme and it is shown in the proofs of Propositions 7 and 8 that for those types who use CBDC under co-existence, we have $q_2>\bar{q}_{w_2}(1)$ and $\gamma_c>1$, respectively.[34]

We have proved that for $\theta=1$, the IC_{12} constraint is not binding. All functions are continuous and it can be shown that the value of the constraint is continuous in θ, therefore, the constraint continues to be slack for θ sufficiently close to 1. This completes the proof.

Case 3. Co-Existence-2 Scheme

We should also show that another scheme in which type 2 buyers use cash and type 1 agents use CBDC, called the co-existence-2 scheme, is not optimal. This case is easy to

[34] Here, I show $q_2>\bar{q}_{w_2}(1)$ independently using the necessary conditions for the planner's problem ignoring the IC_{12}. For that problem, the Lagrangian can be written as follows:

$$(1-\pi_2)f(w_1,\bar{q}_1(\gamma_c))+\pi_2 f(w_2,q_2)-\pi_2\frac{K}{\beta\sigma}+\frac{\lambda}{\beta\sigma\theta}\left(-O(w_2,\gamma_c)+\begin{bmatrix}-(1-\beta)D_{w_2}(q_2)+\beta\sigma\theta(w_2u(q_2)-c(q_2))\\ -K+(1-\pi_2)/\pi_2(\gamma_c-1)D_{w_1}(\bar{q}_1(\gamma_c))\end{bmatrix}\right)$$

FOC with respect to q_2 implies that: $(1+\frac{\pi_2}{\lambda})f_q(w_2,q_2)=\frac{1-\beta}{\beta\sigma\theta}\frac{dD_{w_2}(q_2)}{dq}$. Therefore, q_2 is strictly greater than $\bar{q}_{w_2}(1)$ which is given by: $f_q(w_2,q_2)=\frac{1-\beta}{\beta\sigma\theta}\frac{dD_{w_2}(q_2)}{dq}$.

analyze. Note that under complete information, the co-existence-2 scheme is dominated by the co-existence scheme. With private information and under the conditions specified earlier, welfare under the co-existence scheme is unchanged compared with complete information case. Welfare under the co-existence-2 scheme with private information is weakly lower than that under the co-existence-2 scheme with complete information, because private information simply adds a constraint to the planner's problem. Therefore, with private information, welfare under the co-existence scheme dominates welfare under the co-existence-2 scheme.

□

Proof of Corollary 1. By assumption, there exist a π around which the welfare level is identical under the the CBDC-only scheme or another scheme, say the the cash-only scheme. Given that (14) holds and θ is sufficiently close to 1, we know from Proposition 10 that the IC constraints are binding with private information, so the welfare under the the CBDC-only scheme is strictly less than that under the the cash-only scheme with private information. □

Proof of Preposition 11. If $\alpha_c = 0$, the claim is identical to Proposition 3 proved earlier.

If $\alpha_c > 0$, and under condition (21), the first best can be achieved using the following policy: $\gamma_c = \beta$, γ_e is set sufficiently high ($\gamma_e > 2 - \beta$), and

$$t_e = \begin{cases} (1 - \frac{2-\beta}{\gamma_e})D(q^*) & z_e > (2-\beta)/\gamma_e D(q^*) \\ 0 & \text{otherwise} \end{cases}.$$

□

Proof of Proposition 12. **Proposition 12 (a):**

Suppose by way of contradiction that cash is not valued. I propose the following allocation and show that it achieves higher welfare: $q_e = q^*$, and $z_c = \epsilon > 0$ where ϵ is chosen sufficiently close to 0 that (20) and (19) are satisfied. Since $\alpha_b = 0$, (19) is reduced to

$$-(\gamma_e - \beta)z_e + \beta t_e + \beta\sigma\theta\alpha_e f(q_e) \geq 0.$$

Since in the proposed allocation, $q_e = q^*$, so we must have $z_e + t_e = D(q^*)$. Together with (18), this implies $\gamma_e z_e + (\gamma_c - 1)z_c = D(q^*)$; therefore,

$$-(1-\beta)D(q^*) + \beta\sigma\theta\alpha_e f(q^*) \geq (1-\gamma_c)z_c. \tag{25}$$

By assumption, the LHS is strictly positive, so if z_c is sufficiently small, this constraint is satisfied. Finally, we need to check that $z_c > 0$ is a solution to $\max\{-(\gamma_c - \beta)z_c + \beta\sigma\theta\alpha_c f(q)\}$.

This condition is satisfied if $-(\gamma_c - \beta)(1-\theta) + \beta\sigma\theta\alpha_c > 0$, but γ_c can be sufficiently close to β for this inequality to hold.

In this allocation, the first best is achieved in CBDC meetings, and the production level in cash meetings is strictly positive. Therefore, the welfare level in this allocation is above the welfare level in any allocation in which the production level in cash meetings is zero, and this production level is zero only if cash is not valued.

Now assume that CBDC is not valued in the optimal allocation. Consider the same allocation but with $q_e' = \epsilon$. Equation (25) must hold with ϵ replacing q^*. If $\gamma_c > 1$, then this equation holds because its LHS is positive, following the facts that (21) holds and also $\epsilon < q^*$. If $\gamma_c \leq 1$, since $\gamma_c \geq \beta$ and $z_e < q^*$, then if ϵ is set sufficiently small, this inequality holds following (21).

Finally, it is not possible that neither cash nor CBDC is valued, because any of the allocations valued above yield strictly higher welfare than zero. This completes the proof.

Proposition 12 (b):

In this proof, I assume for simplicity that $\alpha_e = 0$. The proof can be easily extended for $\alpha_e > 0$. I write the LHS of (19) in the following format:

$$\max_{z_c} \big\{-(\gamma_c-\beta)(z_c+z_e+t_e)+\beta\sigma\theta\big(\alpha_c f(q_c)+\alpha_b f(q_b)\big)\big\}+(\gamma_c-\beta)(z_e+t_e)-(\gamma_e-\beta)z_e+\beta t_e(z_e)+\beta\sigma\theta\alpha_e f(q_e).$$

Assuming that $q_b \leq q^*$, we will have $z_c + z_e + t_e = D(q_b)$. Next, using (18), the LHS of (19) can be written as: $\max_{z_c} \big\{-(\gamma_c-\beta)(z_c+z_e+t_e)+\beta\sigma\theta\big(\alpha_c f(q_c)+\alpha_b f(q_b)\big)\big\}+(\gamma_c-1)D(q_b)$.

Now, I write the RHS of (19) as follows:

$$\begin{aligned}
&\max_q \big\{-(\gamma_c-\beta)D(q)+\beta\sigma\theta(\alpha_c+\alpha_b)f(q)\big\}\\
&\geq \max_{z_c} \big\{-(\gamma_c-\beta)(z_c+z_e+t_e)+\beta\sigma\theta(\alpha_c+\alpha_b)f(D^{-1}(z_c+z_e+t_e))\big\}\\
&> \max_{z_c}\{-(\gamma_c-\beta)(z_c+z_e+t_e)+\beta\sigma\theta(\alpha_c f(D^{-1}(z_c))+\alpha_b f(D^{-1}(z_c+z_e+t_e)))\big\}\\
&= \max_{z_c}\{-(\gamma_c-\beta)(z_c+z_e+t_e)+\beta\sigma\theta\big(\alpha_c f(q_c)+\alpha_b f(q_b)\big)\big\},
\end{aligned}$$

where the first inequality is derived by optimality of q and the second one is obtained by the fact that $z_e + t_e > 0$. Altogether, the constraint implies that $(\gamma_c - 1)D(q_b) > 0$, and consequently $\gamma_c > 1$. □

Proof of Proposition 13. The constraint of the planner's problem can be written as:

$$\begin{aligned}
&\max_{z_c} \big\{-(\gamma_c-\beta)z_c+\beta\sigma\theta\big(\alpha_c f(q_c)+\alpha_b f(q_b)\big)\big\}+(\gamma_c-1)z_c-(1-\beta)(z_e+t_e)\\
&\geq \max_q \big\{-(\gamma_c-\beta)D(q)+\beta\sigma\theta(\alpha_c+\alpha_b)f(q)\big\}.
\end{aligned}$$

By way of contradiction, assume that for any $\alpha_b > 0$, CBDC is valued in the optimal allocation. Now, I propose another allocation in which cash inflation and the real CBDC balances in the DM (post-transfers) are slightly lower, but the value of the objective function is higher:

$$\gamma_c' = \gamma_c - \Delta_1,$$

$$z_e' + t_e' = z_e + t_e - \Delta_2.$$

Since $\alpha_e = 0$, the first best cannot be achieved following Proposition 11. If $\gamma_c = \beta$, then buyers can buy the first level of DM production by carrying only cash; therefore, we must have $\gamma_c > \beta$. Also, CBDC is valued, so $z_e + t_e > 0$. Hence, the proposed allocation and policy with γ_c' (and sufficiently high γ_e) are feasible.

I argue here that if α_b is sufficiently small (or, equivalently, α_c is sufficiently large), then sufficiently small values for Δ_1 and Δ_2 can be found that the welfare level is higher in the proposed allocation relative to the original allocation and the constraint continues to be satisfied. If Δ_1 and Δ_2 are small, then one can calculate the changes in the LHS and RHS of the constraint by simply taking a derivative. Hence, we need to show the following:

$$-\left(-z_c+z_c+(\gamma_c-1)D'(q_1)\frac{\partial q_1}{\partial \gamma_c}\right)\Delta_1-\left(\beta\sigma\theta\alpha_b f'(q_b)\frac{\partial q_b}{\partial(z_e+t_e)}+(\gamma_c-1)\frac{\partial z_c}{\partial(z_e+t_e)}-(1-\beta)\right)\Delta_2 \geq D(\tilde{q})\Delta_1,$$

where

$$\tilde{q} \equiv \arg\max_q \big\{ -(\gamma_c-\beta)D(q) + \beta\sigma\theta(\alpha_c+\alpha_b)f(q)\big\}.$$

For writing this inequality, I differentiated both sides with respect to γ_c and $(z_e + t_e)$, and I also used the envelope theorem. After simple algebra, we need to show:

$$\left((1-\beta)-\beta\sigma\theta\underbrace{\alpha_b}_{\text{small}}f'(q_b)\frac{\partial q_b}{\partial(z_e+t_e)}-(\gamma_c-1)\underbrace{\frac{\partial z_c}{\partial(z_e+t_e)}}_{\text{small}}\right)\Delta_2 \geq \left(D(\tilde{q})+(\gamma_c-1)D'(q_1)\frac{\partial q_1}{\partial \gamma_c}\right)\Delta_1.$$

If α_b is set sufficiently small, then the coefficient in the LHS will become positive, because z_c is not affected by much when α_b is small. Therefore, we can find a sufficiently small value for Δ_1/Δ_2 that this inequality holds. The objective function has now increased, since the dominant term is $\alpha_c f(q_c)$ (because α_b is small), and q_c has increased (because cash inflation is lower). This is a contradiction because we could find a feasible solution with a higher value for the objective function. This completes the proof.

Proof of part (b) is similar to (and easier than) part (a), so I skip it. Assumption $u'(0) < \infty$ is needed to ensure that when agents' CBDC balances are reduced, the reduction in the utility level is bounded. □

On the Calibration Exercise

I elaborate on some details of the calibration exercise in Section 6. As a reminder regarding the notation, we have:

$$O(w,\gamma) \equiv \max_q \{-(\gamma-\beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\}, \tag{26}$$

$$\bar{q}(w,\gamma) \equiv \arg\max_q \{-(\gamma-\beta)D_w(q) + \beta\sigma\theta(wu(q) - c(q))\}. \tag{27}$$

Denote

$$i = \frac{\gamma-\beta}{\beta}.$$

One can write the first-order condition for a given inflation rate as:

$$(-i(1-\theta) + \sigma\theta)wu'(q) - (i+\sigma)\theta c'(q) = 0.$$

Assume the following functional forms:

$$u(q) = q^{1-\eta}, c(q) = c_1 q.$$

Hence, $\bar{q}(w,\gamma)$ is given by:

$$\bar{q}(w,\gamma) = \left(\frac{(-i(1-\theta)+\sigma\theta)w(1-\eta)}{(i+\sigma)\theta c_1}\right)^{1/\eta}.$$

Also,

$$z(w,i) \equiv D_w(\bar{q}(w,\gamma)) = (1-\theta)wu(\bar{q}) + \theta c(\bar{q}) = \bar{q}((1-\theta)w\bar{q}^{-\eta} + \theta c_1)$$
$$\Rightarrow z(w,i) = \bar{q}(w,\gamma)\theta c_1\left(\frac{(i+\sigma)(1-\theta)}{(-i(1-\theta)+\sigma\theta)(1-\eta)} + 1\right).$$

Price over marginal cost in the DM is thus given by

$$\mu(w,i) = \frac{z(w,i)}{\bar{q}(w,\gamma)c_1} = \frac{(i+\sigma)(1-\theta)}{(-i(1-\theta)+\sigma\theta)(1-\eta)} + 1.$$

Note that $\mu(w,i)$ is independent of w.

The production function in the CM was assumed to be linear in the benchmark model. However, with the linear function, the level of production in the CM is indeterminate. To eliminate this indeterminacy, and following the literature, I assume for this empirical exercise that the production function in the CM is $U(X) = Aln(X)$. This implies that the level of production in the CM is $X^* = A$. The parameters (η, A, θ) are estimated using the standard method developed by Lucas (2001). The idea is to use the relationship between the nominal

interest rate and money demand; i.e., $L(i) = M/(PY)$. Assume the population is composed of two types, w_1 and w_2. In the benchmark model,

$$\hat{L}(i) = \frac{M/P}{Y} = \frac{\pi_1 z_1(w_1, i) + \pi z_2(w_1, i)}{A + \sigma(\pi_1 z_1(w_1, i) + \pi z_2(w_2, i))} = \left(\frac{A}{\pi_1 z_1(w_1, i) + \pi z_2(w_2, i)} + \sigma \right)^{-1}. \quad (28)$$

The parameters are estimated by minimizing the distance between the data- and model-generated money demand. Hence, the following problem is solved:

$$\min_{(\eta, A, \theta)} \quad \sum_{t=1}^{T} (L(i_t) - \hat{L}(i_t))^2 \quad (29)$$

$$\text{s.t.} \quad \pi_1 \mu(w_1, i_0) + \pi \mu(w_2, i_0) = \mu_0, \quad (30)$$

where $L(i_t)$ denotes the $M/(PY)$ from the data at time t. $\hat{L}(i_t)$ is calculated from (28) using the nominal interest rate at time t. The markup at 2% inflation rate (or $i_0 \approx 5\%$) is set to $\mu_0 = 1.20$. The difference between my approach and that typically used in the literature is that I place this constraint explicitly in the minimization problem.

I use M1 to represent M, nominal GDP to represent PY, and the rate on the three-month T-bill to represent i. For the US, the data span 1900 to 2000 and are taken from Craig and Rocheteau (2006). They use the same data set that Lucas used; however, Lucas had included data until only 1994, so Craig and Rocheteau (2006) extended the data set to include data up to 2000. For Canada, the data are from CANSIM. I use series v41552795 for M1 (which includes currency outside banks, chartered bank chequable deposits, less inter-bank chequable deposits). I use series v646937 for nominal GDP. The rate on the three-month T-bill is taken from Table 176-0043. The data span 1967 to 2008. I did not include earlier dates for Canada, because the M1 data series was discontinued and there were some inconsistencies between earlier versions and the current one.

It is worth mentioning that the length of data used here for Canada is short, so a longer data horizon would provide more reliable estimates. The problem with inclusion of data from the distant past is that it is likely that the underlying parameters of the environment become different from the parameters of the environment now, so the estimates may not be useful for any counter-factual analysis; e.g., the estimated cost of 10% inflation. This is not only a general point that may be true in many empirical settings, but it is specifically true in this economy as $M/(PY)$ has decreased from the pre-1980 period to the post-1980 period, because of extensive usage of electronic means of payment (like credit cards) and less demand to hold currency or its equivalents for transaction purposes.