More on Money Mining and Price Dynamics: Competing and Divisible Currencies*

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Abstract

We develop a random-matching model to study the price dynamics of divisible monies produced privately by using a time-consuming mining technology. There exists a unique equilibrium where the value of money increases until it reaches a steady state. There is also a continuum of perfect-foresight equilibria indexed by the starting value of the currency where the price of money inflates and bursts over time. Early on private money is held for a speculative motive and it acquires a transactional role when money supply becomes sufficiently abundant. We study gold- and crypto-mining technologies, fiat and commodity monies, single and competing currencies.

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1 Introduction

Choi and Rocheteau (2019) — CR thereafter — develops a search-theoretic model of a pure currency economy where money is produced privately according to a time-consuming mining technology. They characterize the time-paths of the money supply, prices, and output starting from a zero money supply. They show the existence of a continuum of equilibria, indexed by the initial value of money, where the price of money and its supply increase in the early stage despite money not being used as a means of payment. While the equilibrium with the highest initial value for money reaches a monetary steady state, all other equilibria feature an initial boom followed by a bursting phase where the value of money declines and vanishes asymptotically.

CR is based on Shi (1995) and Trejos and Wright (1995) where money is indivisible and money holdings are restricted to $\{0, 1\}$. While these assumptions provide tractability to study dynamic equilibria in a continuous-time setting, they restrict the possibility to consider competing currencies with different characteristics and growth rates. In this paper, we relax these assumptions and study a continuous-time economy with divisible monies that can be traded either in pairwise meetings or in competitive exchanges, as in Choi and Rocheteau (2020).[1] There are several advantages in using a continuous-time version of the Lagos and Wright (2005) model. First, it is easier to study dynamic systems composed of differential equations in continuous time. Second, the use of continuous time purifies the equilibrium set from exotic dynamics that only prevail when time is discrete.[2] Third, the market for gold and (crypto-)currencies are opened around the clock, and thus a continuous-time model is a good approximation of these markets.[3] We characterize the set of equilibria under various assumptions for mining technologies and properties of competing monies.

We start by considering an economy with a single money as in CR. The mining technology is such that the mining rate decreases with the stock of money already produced. The cost of mining is an opportunity cost associated with an occupation choice. If money is fiat (i.e., it has no intrinsic value), then the results in CR hold: there is a continuum of equilibria where the economy features a boom/bust cycle and the value of money vanishes asymptotically. The peak for the price of money is rank ordered according to its initial value across equilibria. Agents strictly prefer mining to producing early on. When the money supply

[1] See Lagos et al. (2017) for a thorough presentation of different generations of search-theoretic models of monetary exchange.

[2] The equilibrium set of the Lagos and Wright model depends crucially on the length of a period — if the length of a period is sufficiently long then there might exist cycles or exotic dynamics. Choi and Rocheteau (2020) show that our continuous-time model is the limit of the discrete-time Lagos and Wright model as the length of each period vanishes. Hence it approximates markets that open at a high frequency.

[3] Nowadays trading gold is easy, one could go to brick-and-mortar stores or online bullion dealers. In some locations in the US, Europe and Middle East there are vending machines of gold products. Similarly, many Bitcoin exchanges, such as Coinbase and Bittrex, operate continuously. In the US, buying Bitcoin is almost instantaneous and selling could take up to 1-2 days.

reaches a threshold, money starts being used as a medium of exchange and productions take place. There are differences relative to CR: for instance, changes in the mining efficiency or the maximum potential money supply affect the actual supply of money but not the real allocations in the long run. If money takes the form of a Lucas tree that pays a dividend flow, then the equilibrium is unique and there exists a new type of equilibrium where the economy reaches the first-best allocation in the long run and the price of money is constant over time. This regime necessitates that the intrinsic value of money and the potential money supply are large enough to make the endogenous provision of liquidity sufficiently abundant.

We study an alternative mining technology, similar to the one for crypto-currencies, where the money growth rate is determined ex ante independently from the number of miners and the revenue from money creation is divided evenly among miners. In accordance with our benchmark model, the market capitalization of the currency increases over time. However, the price of the currency declines as its supply increases. This last result can be over-turned if the acceptability of the new currency increases over time.

The second part of the paper investigates economies with two monies. Our first economy has two interest-bearing monies (or two commodity monies) that can be produced according to two distinct mining technologies. The two monies coexist in the long run provided that their potential supplies and the rate at which they are mined are not too far apart. The transition to the long-run equilibrium features an economy with a single money in an initial phase followed by dual-money regime in a second phase where both monies are produced and exchanged. If the production rate of one money increases, then the value of both monies decreases. If the second money is introduced after the steady state with one money has been reached, then the value of the two monies falls initially and increases afterwards. Hence, the discovery or invention of new monies can generate price fluctuations.

The second economy has two fiat monies, one produced by the government and one produced privately. Each money is only accepted in a fraction of all meetings. We characterize equilibria where the monetary authority targets the aggregate value of its own currency. We show that the two monies can coexist provided that the rate of return of the government money is not too high and the fraction of matches where only the private money can serve as means of payment is not too low. Along the transitional path to a long-run equilibrium where both monies coexist, the value of the private money increases over time.

Related literature We extend the money mining model Choi and Rocheteau (2019) to have divisible assets in a continuous-time version of the Lagos and Wright (2005) model. The privately-produced asset is either a fiat money or a Lucas tree, as in Geromichalos et al. (2007) and Lagos (2010). The supply of assets is endogenous as in Lagos and Rocheteau (2008), Rocheteau and Rodriguez (2014), and Geromichalos and Herrenbrueck (2018), among others. Relative to these papers, we emphasize the creation of assets through an occupation choice and explicitly formalize a time-consuming mining technology. Branch et al. (2016) also have private provision of liquidity and an occupation choice but the focus is different: the privately-produced asset takes the form of homes produced by Pissarides firms and the occupation choice is made by unemployed workers who can either be in the construction sector or the consumption-good sector. The study of dynamic equilibria in this class of models includes Lagos and Wright (2003) and Rocheteau and Wright (2013). The continuous-time assumption allows us to eliminate some exotic dynamics, such as cycles or chaotic dynamics, as shown by Oberfield and Trachter (2012) and Choi and Rocheteau (2020). Berentsen (2006) is the first to study the private provision of fiat currency in the context of a search-theoretic model with divisible money.

Fernandez-Villaverde and Sanches (2019) study currency competition in the Lagos-Wright model with a fix measure of entrepreneurs who can issue money at some exogenous cost. They show that the existence of a stationary equilibrium depends on the shape of the cost function of issuing money. In our model mining is a time consuming activity and both the measure of miners and the cost of mining are endogenous. Moreover, they focus on stationary equilibria while we study price dynamics. Our extensions with two competing assets are related to Zhang (2014) and Gomis-Porqueras et al. (2017). Both papers focus on eliminating the indeterminacy of the nominal exchange rate in dual currency economies. Our approach to pin down the exchange rate between privately-produced and government monies is closer to Zhang (2014), which in turn follows Lester et al. (2012). Schilling and Uhlig (2019a) consider the coexistence and exchanges of government money and Bitcoin in a stochastic endowment economy and show the exchange rate between Bitcoin and fiat money is a martingale. Schilling and Uhlig (2019b) study a deterministic version of the same endowment economy and derive sufficient conditions such that currency exchanges arise or do not arise in equilibrium. Lotz and Vasselin (2019) develop a New Monetarist model to study the coexistence of fiat and E-money.

2 Environment

The environment is a continuous-time version of the New Monetarist model (Lagos and Wright, 2005; Rocheteau and Wright, 2005).[4] Time is continuous and indexed by $t \in \mathbb{R}_+$. The economy is composed of two types of infinitely-lived agents: a unit measure of buyers and a unit measure of sellers. There are two types of perishable goods: a good $c \in \mathbb{R}$ that is traded in an on-going competitive market and is taken as the numéraire, and a good $q \in \mathbb{R}_+$ exclusively produced and consumed in pairwise meetings. The labels *buyer* and *seller* refer to agents' role in pairwise meetings.

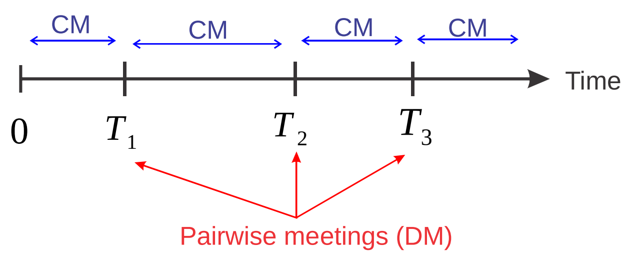

Figure 1: Timing of the model

The lifetime expected discounted utility of buyers is

$$\mathcal{U}^b = \mathbb{E}\left\{\sum_{n=1}^{+\infty} e^{-\rho T_n} u\left[q(T_n)\right] + \int_0^{\infty} e^{-\rho t} dC(t)\right\}, \tag{1}$$

where $C(t)$ is a measure of the cumulative net consumption of the numéraire good.[5] Negative net consumption is interpreted as production. If consumption (or production) of the numéraire happens in flows, then $C(t)$ admits a density, $dC(t) = c(t)dt$. If the buyer consumes or produces a discrete quantity of the numéraire good at some instant t, then $C(t^+) - C(t^-) \neq 0$. The first term between brackets on the right side of (1) accounts for the utility of consumption in pairwise meetings, while the second term accounts for the utility of consuming, or producing if $dC(t) < 0$, the numéraire good. The process $\{T_n\}$ is Poisson with arrival rate $\alpha > 0$, and indicates the times at which the buyer is matched bilaterally with a seller. The utility from consuming q units of goods in pairwise meetings is $u(q)$, where u is strictly concave, $u(0) = 0$, $u'(0) = +\infty$, and $u'(\infty) = 0$.

[4] Choi and Rocheteau (2020) provides a detailed description of the New-Monetarist model in continuous time, its methodology, and applications.

[5] A similar cumulative consumption process is assumed in the continuous-time models of OTC trades of Duffie et al. (2005).

The lifetime expected utility of a seller is

$$\mathcal{U}^s = \mathbb{E}\left\{ -\sum_{n=1}^{+\infty} e^{-\rho T_n} q(T_n) + \int_0^{\infty} e^{-\rho t} dC(t) \right\}.$$

The first term corresponds to the linear disutility of producing q in pairwise meetings. The second term is the discounted sum of utility from the consumption of the numéraire good.

Buyers and sellers cannot access the technology to produce the numéraire good while in pairwise meetings. Moreover, unsecured promises to repay loans are not credible due to lack of commitment and monitoring. These assumptions imply that the buyer of the good in pairwise meetings cannot finance q with the production of the numéraire, thereby creating a need for a means of payment.

Money takes the form of a Lucas tree that pays a dividend flow $d \geq 0$ in numéraire good. The case $d = 0$ corresponds to fiat money. Money is perfectly storable and durable. The quantity of money in the economy at time t is denoted A_t. The price of money in terms of the numéraire is denoted ϕ_t. Money is produced privately according to a time consuming activity called mining. The individual effort devoted to mining by agent i is denoted $e_i \in \{0, 1\}$ and is interpreted as an indivisible occupation choice.[6] If $e_i = 1$ in some time interval around t, then agent i cannot produce when he meets a buyer in that time interval. The aggregate mining effort is $m_t = \int_0^1 e_{i,t} di$ and is also the measure of miners. Given the effort $e = 1$, an agent mines a unit of money according to a Poisson process with time-varying intensity $\lambda(\bar{A} - A_t)$ where $\bar{A}$ is the overall fixed quantity of money, $\bar{A} - A_t$ is the amount of money that has yet to be mined, and $\lambda > 0$ is the productivity of the mining technology. With that specification, the individual mining rate declines as the quantity of money that has been mined, A_t, increases. The individual mining rate, however, is unaffected by the aggregate mining intensity, i.e., the congestion effect from other miners is only indirect through A_t. Later in the paper, we consider an alternative mining technology where the individual mining rate depends negatively on the aggregate mining intensity.

The flow cost of mining is the opportunity cost from mining instead of producing consumption goods. This opportunity cost is endogenous and depend on the quantity and price of money in the economy. To account for the effect of the mining choice on trade probabilities, we denote χ_t the probability with which a seller is available for production, i.e., he is not mining.

[6] In Choi and Rocheteau (2019) we consider an indivisible money model with a more general mining technology. We allow the effort of mining $e_i \in \mathcal{E} = R_+$ where e_i is a variable input such as CPU time or and electricity. We consider gold and crypto mining technology as well as endogenous and exogenous cost of mining. See also Subsection 3.3 for a succinct discussion of the crypto-mining technology.

3 Single-money economies

We start by studying economies with a single money. We will define equilibria and characterize their full set and price dynamics.

3.1 Equilibrium

Equilibria are defined in terms of Bellman equations for the value functions of buyers and sellers, bargaining outcomes in pairwise meetings, optimal choices of real balances and occupation, and market clearing in centralized markets.

Value functions Let $V^b(a)$ be the value function of a buyer with a units of real balances expressed in terms of the numéraire. At any point in time between pairwise meetings, the buyer adjusts her asset holdings to some targeted level, a_t^*, by consuming or producing the numéraire good. The value function from entering a match with a_t^* units of real balances is $W_t^b(a_t^*)$. In Choi and Rocheteau (2020) we show that $V_t^b(a)$ solves the following Hamilton-Jacobi-Bellman (HJB) equation,

$$\rho V_t^b(a) = \max_{a_t^* \geq 0} \left\{ \rho (a - a_t^*) + r_t a_t^* + \alpha \chi_t \left[W_t^b(a_t^*) - V_t^b(a_t^*) \right] + \dot{V}_t^b(a) \right\}, \tag{2}$$

where $r_t \equiv (d + \dot{\phi}_t)/\phi_t$ denotes the rate of return of money, and $\dot{V}_t^b(a) \equiv \partial V_t^b(a)/\partial t$. The right side of (2) has the following interpretation. In order to readjust her real balances from a to a_t^* the buyers must produce $a_t^* - a$ units of numéraire at a unit cost, i.e., the flow cost is $\rho (a - a_t^*)$. After readjusting his real balances the buyer enjoys the rate of return on her targeted real balances, $r_t a_t^*$. With Poisson arrival rate α, the buyer meets a seller, and with probability χ_t the seller is active, i.e., he is not mining. In this case the buyer enjoys a capital gain $W_t^b(a_t^*) - V_t^b(a_t^*)$. Since we are not restricting ourselves to stationary equilibria, the last term on the right side captures the change in the buyer's value function over time. It is immediate from (2) that $V^b(a)$, is linear in a, i.e., $V_t^b(a) = a + V_t^b(0)$, due to the linearity of the preferences with respect to the numéraire and the fact that buyers can produce discrete quantities of the numéraire in the competitive market. Moreover, the targeted real balances, a_t^*, are independent of the initial real balances of the buyer.

Assuming $W_t^b(a_t^*)$ is concave, the buyer's optimal choice of real balances is given by the first-order condition of the maximization problem on the right side of (2):

$$-(\rho - r_t) + \alpha \chi_t \left[W_t^{b\prime}(a_t^*) - 1 \right] \leq 0, \quad " = " \text{ if } a^* > 0. \tag{3}$$

The term $\rho - r_t$ is the flow cost of holding real balances: it is the difference between the rate of time preference, which is also the rate of return of an illiquid bond, and the rate of return of money. The second term on the left side is the marginal value of real balances, which is equal to the instantaneous probability of a match times the marginal surplus from a real balances, $W_t^{b\prime}(a_t^*) - 1$.

There is a similar HJB equation for the value function of a seller:

$$\rho V_t^s(a) = \max_{a_t^* \geq 0, e_t \in \{0,1\}} \left\{ \rho\left(a - a_t^*\right) + r_t a_t^* + \lambda(\bar{A} - A_t)e_t\phi_t + \alpha(1-e_t)\left[W_t^s\left(a_t^*\right) - V_t^s\left(a_t^*\right)\right] + \dot{V}_t^s(a) \right\}. \quad (4)$$

We will show later that $W_t^s(a_t^*)$ is linear because sellers have no transactional motive to hold real balances. As a result, $W_t^s(a_t^*) - V_t^s(a_t^*)$ is independent from a_t^* and sellers do not want to hold money if $r_t < \rho$ or they are indifferent between any quantity of money if $r_t = \rho$. Hence, with no loss in generality we assume that they consume the money they hold, $V_t^s(a) = a + V_t^s$ where V_t^s is independent of a and its value will be determined later. The choice of e represents the decision to mine, $e = 1$, or to produce, $e = 0$. The flow payoff from mining is $\lambda(\bar{A} - A_t)\left[V_t^s(\phi_t) - V_t^s(0)\right] = \lambda(\bar{A} - A_t)\phi_t$ since the miner receives one unit of money at Poisson rate $\lambda(\bar{A} - A_t)$. The flow payoff from producing is $W_t^s(0) - V_t^s(0)$, where W_t^s is determined next.

Pairwise meetings We now turn to the bargaining problem in a pairwise meeting between a buyer holding a^b units of assets and a seller holding a^s units of asset. The outcome of the negotiation is a pair $(q, p) \in \mathbb{R}_+ \times [-a^s, a^b]$ where q is the amount of goods produced by the seller for the buyer and p is the transfer of assets from the buyer to the seller. Feasibility requires that $-a^s \leq p \leq a^b$. By the linearity of $V^b(a)$ and $V^s(a)$, the buyer's surplus is $u(q) + V^b(a^b - p) - V^b(a^b) = u(q) - p$ and the seller's surplus is $-q + V^s(a^s + p) - V^s(a^s) = -q + p$.

In order to determine (q, p) we adopt the Kalai bargaining solution according to which the buyer's surplus is equal to $\theta/(1-\theta)$ the seller's surplus and the trade is Pareto efficient. Formally,

$$(q, p) \in \arg\max \{u(q) - p\} \quad \text{s.t.} \quad u(q) - p = \frac{\theta}{1-\theta}(-q + p) \quad \text{and} \quad -a^s \leq p \leq a^b. \quad (5)$$

It is easy to check from the constraint that the payment is equal to $p = \omega(q) \equiv (1-\theta)u(q) + \theta q$. Moreover, the constraint $p \geq -a^s$ is not binding. Hence, we can reduce the bargaining problem to:

$$q \in \arg\max \theta \{u(q) - q\} \quad \text{s.t.} \quad \omega(q) \leq a^b. \quad (6)$$

The buyer chooses his real balances and the associated consumption level so as to maximize the match surplus subject to the feasibility constraint that the payment cannot be greater than the buyer's real balances. The solution, denoted $q(a^b)$, is such that

$$q(a^b) = \begin{cases} q^* & \text{if } \; a^b \geq \omega(q^*) \\ \omega^{-1}(a^b) & \text{otherwise.} \end{cases} \tag{7}$$

If the buyer holds enough real balances to finance q^* then she purchases q^* and the payment is $\omega(q^*)$. Otherwise, the buyer spends all his real balances and her consumption level solves $\omega(q) = a^b$.

We can now write the value functions of the buyer and the seller upon entering a pairwise meeting. The value function of the buyer is given by:

$$W_t^b(a_t^*) = \theta \left\{ u\left[q(a_t^*)\right] - q(a_t^*) \right\} + V_t^b(a_t^*). \tag{8}$$

The buyer enjoys a fraction θ of the match surplus that depends on her own real balances and her continuation value is $V_t^b(a_t^*)$. The marginal value of real balances in a match is

$$W_t^{b\prime}(a_t^*) = \theta \frac{\left\{u'\left[q(a_t^*)\right] - 1\right\}}{\omega'\left[q(a_t^*)\right]} + 1. \tag{9}$$

It can easily be checked that $W_t^{b\prime}(a_t^*)$ is non-increasing in a_t^* and $W_t^{b\prime}(a_t^*) = 0$ for all $a_t^* \geq \omega(q^*)$. The value function of the seller solves

$$W_t^s(a) = (1 - \theta)\left[u\left(q_t\right) - q_t\right] + V_t^s(a), \tag{10}$$

where q_t only depends on the buyer's real balances. Hence, $W_t^{s\prime}(a) = 1$.

Choice of real balances Having determined the value functions of buyers and sellers in pairwise meetings, we are now in position to solve for the optimal choice of real balances. If we substitute $W_t^b(a_t^*)$ given by (8) into (2), the buyer's value function with her optimal real balances solves:

$$\rho V_t^b(a_t^*) = \max_{a_t^* \geq 0} \left\{ r_t a_t^* + \alpha \chi_t \theta \left\{ u\left[q(a_t^*)\right] - q(a_t^*) \right\} + \dot{V}_t^b(a_t^*) \right\}. \tag{11}$$

According to the third term, the buyer receives an opportunity to consume at Poisson arrival rate $\alpha\chi_t$, in which case his match surplus is $\theta\left[u(q) - q\right]$.

Substituting $W_t^{b\prime}(a_t^*)$ given by (9) into (3), the optimal choice of real balances is given by

$$-(\rho - r_t) + \alpha \chi_t \theta \left[\frac{u'\left[q(a_t^*)\right] - 1}{(1-\theta) u'\left[q(a_t^*)\right] + \theta} \right] \leq 0, \quad " = " \; \text{ if } a_t^* > 0. \tag{12}$$

The first term is the cost of holding real balances while the second term marginal is the expected gain from holding real balances. Assuming $u'(0) = +\infty$, the solution is interior if $\rho - r_t \leq \alpha \chi_t \theta / (1 - \theta)$.

Optimal mining We now substitute W_t^s from (10) into (4), to obtain the following HJB equation for the seller:

$$\rho V_t^s = \max_{e_t \in \{0,1\}} \left\{ \lambda(\bar{A} - A_t)e_t\phi_t + \alpha(1-e_t)(1-\theta)\left[u\left(q_t\right) - q_t\right] + \dot{V}_t^s \right\}. \tag{13}$$

According to the first term in the right side, a seller who chooses to mine is rewarded with one unit of money worth ϕ_t at Poisson arrival rate $\lambda(\bar{A} - A_t)$. According to the second term, a seller meets a buyer with Poisson arrival rate α. If he does not mine, he produces q_t and enjoys a fraction $1-\theta$ of the total surplus.

By (13) the choice of optimal mining intensity solves

$$e_t^* \in \arg\max_{e_t \in \{0,1\}} \left\{ \lambda(\bar{A} - A_t)e_t\phi_t + \alpha(1-e_t)(1-\theta)\left[u\left(q_t\right) - q_t\right] \right\}. \tag{14}$$

An agent who chooses to mine gives up its opportunities to produce, but agents can move freely between occupations. Allowing for asymmetric choices by sellers, the aggregate measure of miners is $m_t \equiv \int_0^1 e_{i,t}^* di$. From (14) it solves

$$m_t \begin{cases} = 1 \\ \in [0,1] \\ = 0 \end{cases} \quad \text{if} \quad \lambda(\bar{A} - A_t)\phi_t \begin{cases} > \\ = \\ < \end{cases} \alpha(1-\theta)\left[u\left(q_t\right) - q_t\right]. \tag{15}$$

The probability that a seller chosen at random in the population is able to produce is $\chi_t = 1 - m_t$.

The law of motion for the supply of money in circulation in the economy is:

$$\dot{A} = m_t \lambda(\bar{A} - A_t). \tag{16}$$

Given the total measure of miners, m, money creation is $m\lambda(\bar{A} - A_t)$. By market clearing:

$$a_t^* = \phi_t A_t. \tag{17}$$

From (12) at equality, the value of money evolves according to

$$\frac{\dot{\phi}_t + d}{\phi_t} = \rho - \alpha\chi_t\theta\left[\frac{u'\left[q(\phi_t A_t)\right] - 1}{(1-\theta)u'\left[q(\phi_t A_t)\right] + \theta}\right]. \tag{18}$$

The value of money solves a first-order, nonlinear differential equation that states that the rate of return of money is equal to the rate of time preference less a liquidity premium that depends on the total capitalization of the money stock.

We are now ready to define an equilibrium as a sequence of time-paths for targeted real balances, measure of miners, value of money, and money supply that solves a system of differential equations.

Definition 1 *A single-currency equilibrium is a list, $\langle a_t^*, m_t, \phi_t, A_t \rangle$, that solves (15), (16), (17), (18) and the initial condition A_0.*

3.2 Price dynamics

We characterize the set of equilibria and price dynamics for economies with an interest-bearing money ($d > 0$) and economies with fiat money ($d = 0$). From (15) and (17) the locus of pairs (A, ϕ) such that agents are indifferent between mining and producing is given by:

$$\lambda(\bar{A} - A)\phi = \alpha(1 - \theta)S(A\phi), \tag{19}$$

where $S(A\phi) \equiv u(q) - q$ is the total surplus from a pairwise trade and quantities solve $\omega(q) = \min\{\omega(q^*), \phi A\}$. For all $A > \lambda\bar{A}/(\alpha + \lambda)$, there is a unique $\phi > 0$ solution to (19). In the (A, ϕ) space, this solution is upward-sloping, features $A > 0$ when $\phi = 0$, and has a vertical asymptote, $A = \bar{A}$. To the left of this locus all agents choose to mine, $m = 1$.

By (18) the locus of stationary equilibria where $m = 0$ and $\dot{\phi} = 0$ is given by

$$\rho = \frac{d}{\phi} + \alpha\theta \left\{ \frac{u'(q) - 1}{(1 - \theta)u'(q) + \theta} \right\}, \tag{20}$$

where q is an implicit function of ϕA. It is a standard asset pricing equation where the effective rate of return of the asset is composed of the pecuniary return, d/ϕ, and the liquidity return. It gives a negative relationship between ϕ and A with $\phi = d/\rho$ for all A such that $A \geq \rho\omega(q^*)/d$. The two loci, (19) and (20), represented in the phase diagram of Figure 2, allow us to characterize stationary and non-stationary equilibria.

Proposition 1 ***(Mining divisible assets)*** *Suppose* $A_0 = 0$.

1. ***(Abundant liquidity)*** *If*

$$\frac{\bar{A}d}{\rho} \geq \frac{\alpha(1 - \theta)\left[u(q^*) - q^*\right]}{\lambda} + \omega(q^*), \tag{21}$$

then there exists a unique equilibrium and it is such that $\phi_t = d/\rho$, $q_t = q^*$ *for all* t, *and* $m_t = \mathbf{1}_{\{t<T\}}$ *where*

$$T = \ln\left\{ \frac{\lambda d\bar{A}}{\rho\alpha(1 - \theta)\left[u(q^*) - q^*\right]} \right\}^{\frac{1}{\lambda}} < +\infty.$$

Moreover,

$$\begin{aligned} A_t &= \bar{A}\left(1 - e^{-\lambda t}\right) \quad \textit{for all } t < T \\ A_t &= A^s = \bar{A} - \frac{\rho\alpha(1 - \theta)\left[u(q^*) - q^*\right]}{\lambda d} \quad \textit{for all } t \geq T. \end{aligned}$$

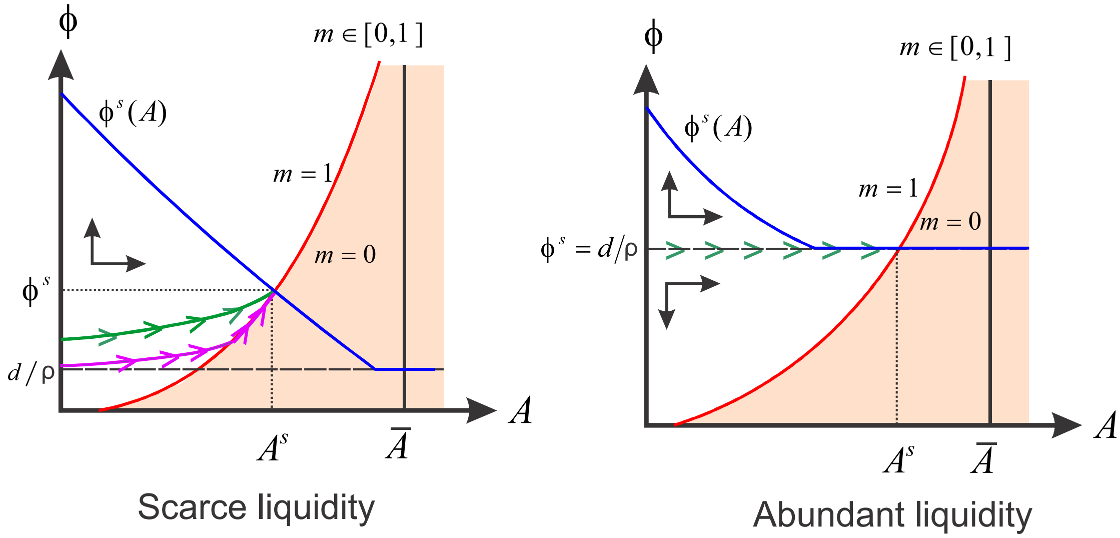

Figure 2: Phase diagram with divisible assets

2. ***(Scarce liquidity)*** *Assume u'' is bounded. If $d > 0$ and (21) does not hold, then there exists a unique equilibrium and it is such that $\phi_t > d/\rho$ and $q_t < q^*$ for all t. Moreover, $m_t = 1$ for all t sufficiently small and the economy transitions to $m_t < 1$ before reaching the steady state if*

$$\frac{\theta}{1-\theta} > \frac{\nu(A^s\phi^s) + A^s/(\bar{A} - A^s)}{\nu(A^s\phi^s)\left[1 - \nu(A^s\phi^s)\right]} \tag{22}$$

where $\nu(x) \equiv S'(x)x/S(x)$. If the elasticity $u'(q)q/u(q)$ falls in q, then there is at most one transition for all $t \geq 0$, and it is from $m_t = 1$ to $m_t < 1$.

3. ***(Fiat money)*** *If $d = 0$, then there exists a unique equilibrium leading to a monetary steady state if $\rho < \alpha\theta/(1-\theta)$. The equilibrium is such that $q_t < q^*$ and $\dot{\phi}_t \geq 0$ for all t and the steady state solves*

$$\rho = \alpha\theta \left\{ \frac{u'(q^s) - 1}{(1-\theta)u'(q^s) + \theta} \right\} \tag{23}$$

$$\frac{A^s}{\bar{A}} = \frac{\lambda\omega(q^s)}{\alpha(1-\theta)\left[u(q^s) - q^s\right] + \lambda\omega(q^s)}. \tag{24}$$

There is a continuum of boom-burst equilibria indexed by ϕ_0 such that: $\dot{\phi}_t > 0$ for all $t < T(\phi_0)$ where $T(\phi_0) > 0$; $\dot{\phi}_t < 0$ for all $t > T(\phi_0)$; $\lim_{t \to +\infty} \phi_t = 0$.

Proposition 1 distinguishes two regimes. First, there is a regime where the asset supply at the steady state is abundant enough to satiate agents' liquidity needs and allow agents to trade q^* in all matches. In such equilibria, the asset is priced at its fundamental value, $\phi = d/\rho$, at all dates. A necessary (but not sufficient) condition is that the potential asset supply when valued at its fundamental price, $\bar{A}d/\rho$, is larger than agents' liquidity needs, $\omega(q^*)$. It is the standard condition in the literature for abundant liquidity since Geromichalos et al. (2007), except that it applies to the potential asset supply, $\bar{A}$, and not the actual asset supply, A, which is endogenous. Condition (21) has an extra term that captures agents' incentives to mine. This term falls in λ so liquidity is more likely to be abundant when the mining speed is large. In equilibria with abundant liquidity, $q_t = q^*$ even before market capitalization has reached $\omega(q^*)$. The reason for this result is that as long as the steady-state asset supply has not been reached, all agents mine and hence there is no demand for liquidity. When the steady state is reached, in finite time, then agents start trading. Finally, the money supply solves a first-order, linear differential equation with constant term that is solved in closed form.

The second regime with scarce liquidity is qualitatively similar to the one in CR in a model with indivisible money. The price of the asset is above its fundamental value at all dates and it keeps rising over time until it reaches a steady state. So our model predicts a positive correlation between the value of money and its supply along the transitional dynamics to the steady state. In contrast, in the long run, if λ or $\bar{A}$ increases, then A^s increases but ϕ^s decreases. So, while the correlation between A and ϕ is opposite to the quantity theory along transitional dynamics, it is consistent with it across steady states.

In both types of equilibria, sellers specialize in mining early on. Hence, there is no bilateral trade and the velocity of money is zero. From (18), $\dot{\phi}_t = \rho\,(\phi_t - d/\rho)$, i.e., the difference between the market value of money and its fundamental value grows at rate ρ. In that initial phase, money looks like a speculative bubble. Agents start using money as a means of payment only when the supply of money becomes sufficiently high, in which case the velocity of money rises and its rate of appreciation falls.[7]

Part 2 of Proposition 1 distinguishes two distinct ways to reach the steady state when liquidity is scarce. In one case, miners and producers coexist in the neighborhood of the steady state, i.e., $m_t \in (0,1)$, the velocity of money increases gradually, and the steady state is reached only asymptotically. In the second case, all sellers strictly prefer mining, $m_t = 1$, until the steady-state is reached in finite time. The velocity of money varies discontinuously from 0 to α. According to condition (22), the first case prevails if the buyer's bargaining share is sufficiently large. These two cases are illustrated in Figure 2 as a purple line and a green line respectively.

[7] The result that the velocity of money is zero initially depends on the matching technology. In the Online Appendix we show that if we use a general matching function, then there will be trade throughout the equilibrium path.

From (23), in the case where the asset is a pure fiat money, $d = 0$, then q^s is independent of $\bar{A}$ and λ. The reason why $\bar{A}$ and λ do not matter for real allocations is because money is neutral, hence the endogenous stock of money at the steady state is irrelevant. From (24), the endogenous money supply is proportional to the potential supply, where the coefficient of proportionality increases with λ. While the equilibrium with $d > 0$ is unique, there is a continuum of equilibria when $d = 0$. In addition to the unique equilibrium leading to the steady state, there is a continuum of boom/burst monetary equilibria indexed by the initial value of money where the value of money increases and then decreases and goes to zero asymptotically. So the higher the initial value of money, ϕ_0, the higher the value of money at all points in time. For all equilibria except the highest one, the value of money increases initially even though agents anticipate that it will ultimately collapse, we illustrate these trajectories in the left panel of Figure 3.

By (23) and (24), the condition (22) can be rewritten as:

$$\frac{\rho(1-\theta)+\theta\lambda}{\rho\theta} < \frac{\alpha\theta\left[u(q^s)-q^s\right]-\rho\omega(q^s)}{\alpha\theta\left[u(q^s)-q^s\right]}.$$

It does not hold when r is close to 0, i.e., agents specialize in mining until the steady state is reached, and it reduces to $\theta\lambda < \rho(2\theta - 1)$ when α goes to $+\infty$. The fact that this condition depends on λ shows that even though λ does not affect the steady-state equilibrium, it does matter for the transition leading to it.

The next corollary shows changes in $\bar{A}$ are neutral in the long run but have real effects along the transitional path.

Corollary 1 ***(Money neutrality)*** *Assume $d = 0$. An increase in $\bar{A}$ leads to a proportional increase in A^s in the long run and no real effects. In the short run, aggregate real balances and output fall. During the transition to the new steady state, the inflation rate is negative.*

In the right panel of Figure 3 we illustrate a phase diagram where $\bar{A}$ rises from $\bar{A}_1$ to $\bar{A}_2$ and then to $\bar{A}_3$. Although the price of money fluctuates over time, the long run aggregate real balances $Z^s = \phi A$ remain unchanged.

3.3 Crypto mining

We now describe succinctly how our results could vary depending on the formalization of the mining technology. In our benchmark specification, the money growth rate is endogenous and depends on the mining activity. For several crypto-currencies (e.g., Bitcoin) the aggregate rate of money creation does not vary with

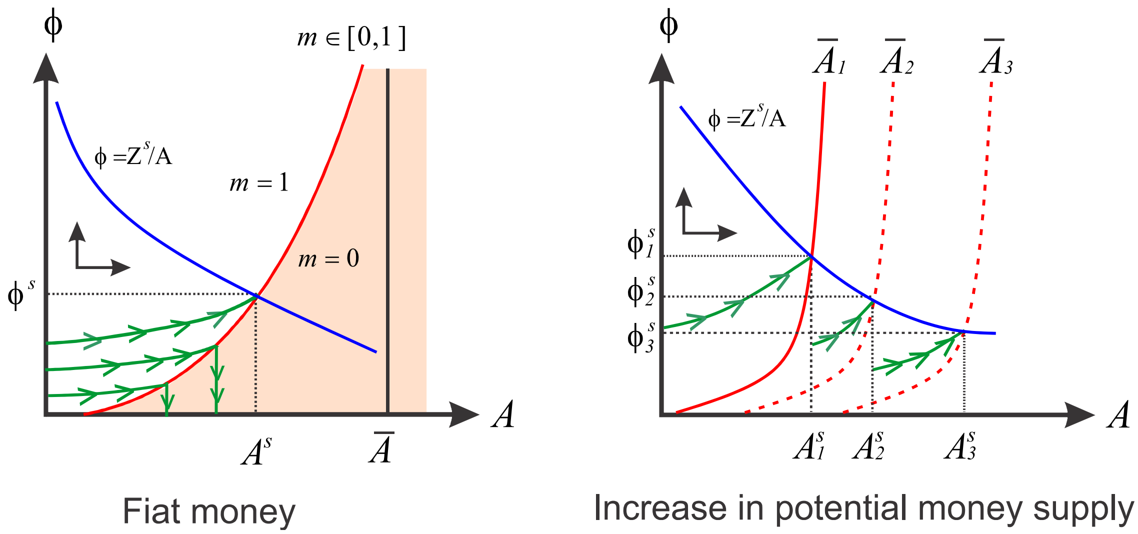

Figure 3: Phase diagram with fiat money

the measure of miners. The designer of the fiat currency chooses a path for the money supply, $\dot{A}_t = \pi(A_t)A_t$, where $\pi(A_t)$ is the state-contingent rate of money creation. The seigniorage revenue, $\pi(A_t)A_t$, is exogenous and independent of the measure of miners. Each unit of newly created money is allocated to a miner with a probability proportional to their mining effort. Moreover, suppose that mining is no longer an occupation choice. Sellers choose a mining effort $e \in \mathbb{R}_+$ at some unit cost k and can keep on producing when they meet buyers. The aggregate mining effort is $m_t = \int e_t^i di$. Again, we denote $Z \equiv \phi A$ the aggregate real balances.

The HJB equation for the value function of a seller is now:

$$\rho V_t^s = \max_{e_t \in \mathbb{R}_+} \left\{ -e_t k + \frac{e_t}{m_t} \pi_t Z_t + \alpha(1-\theta)\left[u\left(q_t\right) - q_t\right] + \dot{V}_t^s \right\}. \tag{25}$$

The miner exerts effort e_t to receive a part of the money creation equal to $e_t \pi_t Z_t / m_t$. Due to the linearity of the cost and benefit of mining, miners are indifferent between mining or not in equilibrium. The first-order condition with respect to e_t is

$$m_t = \frac{\pi_t Z_t}{k}. \tag{26}$$

The equilibrium measure of miners is equal to the seigniorage revenue divided by the cost of mining.

The HJB equation for the value function of a buyer and the optimal choice of real balances solve (11) and (12) with $\chi_t = 1$. In a monetary equilibrium where the buyer's optimality condition holds at equality,

(A, Z) satisfies the following ODEs:

$$\frac{\dot{Z}}{Z} = \rho + \pi(A) - \alpha\theta \left[\frac{u'(q) - 1}{(1-\theta)u'(q) + 1} \right] \tag{27}$$

$$\dot{A} = \pi(A)A. \tag{28}$$

Suppose $\pi'(A) < 0$ and $\pi(A) > 0$ for $A < \bar{A}$ and $\pi(A) = 0$ otherwise. Then the Z-isocline ($\dot{Z} = 0$) is upward-sloping while the A-isocline ($\dot{A} = 0$) is vertical at $A = \bar{A}$ as shown in Figure 4. An equilibrium is a time path (A, Z, m) that solves (26), (27), and (28). The equilibrium has a recursive structure. The supply of money, A, is determined independently of Z and m. Given A one can determine Z from (27). And given A and Z one can determine m from (26).

Proposition 2 *Suppose there is $\bar{A} > 0$ such that $\pi(A) > 0$ for all $A < \bar{A}$ and $\pi(\bar{A}) = 0$. There exists a unique equilibrium, Z_t^*, leading to a steady state. If $\pi'(A) < 0$ then $\dot{Z}_t^* > 0$, $\dot{r}_t^* > 0$, and $\dot{\phi}_t^* < 0$. There is also a continuum of equilibria indexed by $Z_0 \in (0, Z_0^*)$ such that $\lim_{t\to\infty} Z_t = 0$. For all equilibria $Z_0 \in (0, Z_0^*)$, there exists $t_0 > 0$ such that $\dot{Z}_t^* > 0$ for all $t \in (0, t_0)$ and $\dot{Z}_t^* < 0$ for all $t > t_0$.*

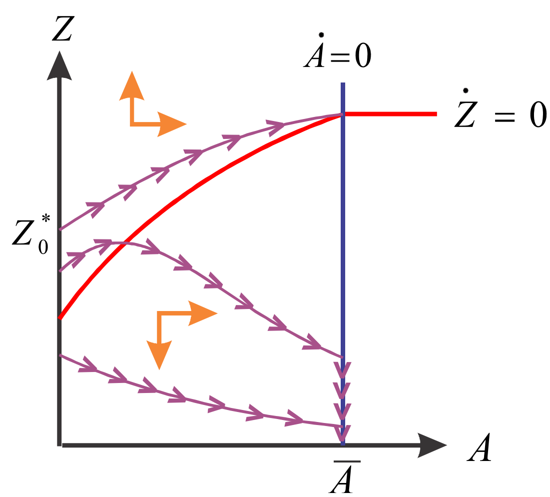

Figure 4: Phase diagram of the crypto-mining model

Proposition 2 shows that there exists a continuum of equilibria indexed by the initial value of aggregate real balances, $Z_0 \in [0, Z_0^*]$. The equilibrium corresponding to $Z_0 = Z_0^*$ is the unique equilibrium that leads to a steady state. Along this equilibrium path, Z_t increases. So the market capitalization of the currency increases as its money supply increases and so does output. From (12) the fact that Z_t increases means that r_t increases so that buyers wish to hold more real balances over time. So the inflation rate decreases over time. Because $r_T = 0$, it follows that $r_t < 0$ for all $t < T$. So inflation is positive throughout the transition leading to the steady state and the price of the currency decreases over time. There are other equilibria, indexed by $Z_0 < Z_0^*$, so that aggregate real balances first increase and then vanishes asymptotically.

In any monetary equilibrium the new currency is universally accepted at all points in time. Because the acceptability of a currency is critical for its price dynamics, we now introduce it explicitly in the model. We assume α_t is exogenous and time varying. We interpret α_t as the product of a matching arrival rate and the probability to accept money conditional on a match being formed. In order to disentangle the effects from money growth and acceptability on currency prices, we assume $\pi_t = \pi$ is constant. We make the following additional assumptions: $\dot{\alpha}_t > 0$ for all $t < T$ and $\dot{\alpha}_t = 0$ for all $t \geq T$. So acceptability increases over time.

The ODE for aggregate real balances satisfies:

$$\frac{\dot{Z}_t}{Z_t} = \rho + \pi - \alpha_t \left[\frac{u'(q_t) - 1}{(1-\theta)u'(q_t) + \theta} \right]. \tag{29}$$

We can treat t as a state variable with the law of motion $\dot{t} = 1$ and plot the system of ODEs for (Z_t^*, t) in two-dimensional phase diagrams in Figure 5. In the left panel acceptability is constant through time, $\alpha_t = \alpha$. The Z-isocline is horizontal and corresponds to the steady-state aggregate real balances, Z^s. The unique equilibrium that becomes stationary at some point in the future is such that $Z_t^* = Z^s$ for all t. Hence, aggregate real balances are constant and the rate of price changes is determined by the money growth rate.

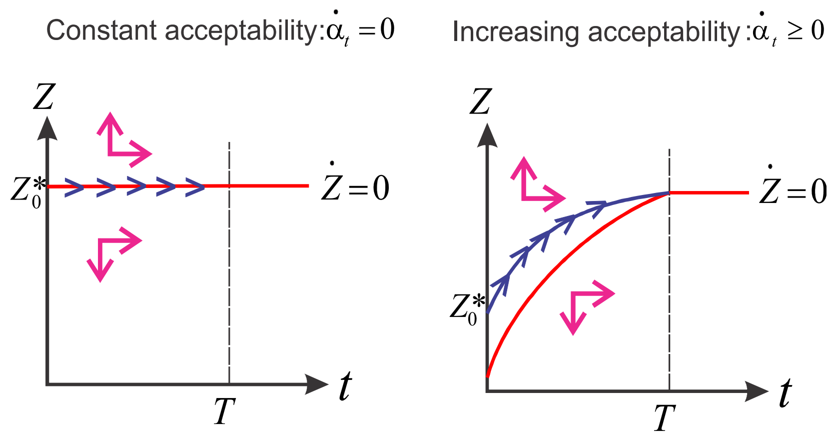

Figure 5: Dynamics of aggregate real balances under time-varying acceptability

In the right panel of Figure 5 we represent the case where acceptability increases over time until it becomes constant after T. The Z-isocline is now upward-sloping since steady-state Z increases with α_t. The unique equilibrium leading to the steady state after T is such that aggregate real balances increase over time.

If $\pi = 0$, the value of money increases over time due to the increase in acceptability. We summarize these results in the following proposition.

Proposition 3 ***(Time-varying acceptability).*** *Suppose $\dot{A}/A = \pi$ for all t, $\dot{\alpha}_t \geq 0$ for all $t \leq T$, and $\dot{\alpha}_t = 0$ for all $t > T$. There exists a unique monetary equilibrium such that $Z_t^* = Z^s$ for all $t \geq T$ and it is such that $\dot{Z}_t^* > 0$ for all $t < T$ if $\dot{\alpha}_t > 0$ for all $t < T$. If $\dot{\alpha}_t = 0$ for all t, then $Z_t^* = Z^s$ for all t.*

4 Dual money economies

We now turn to economies with multiple monies. We first consider the case of two interest-bearing monies with two mining technologies. We characterize the production of the two monies and the evolution of their prices along a transitional path to a steady state. Second, we study an economy with two fiat monies, one government money and one private money. In both cases, our assumptions are chosen to avoid the indeterminacy of the exchange rate between competing monies (Kareken and Wallace, 1981).

4.1 Competing interest-bearing monies

Suppose that there are two Lucas trees that can serve as means of payment, silver (A^g) and gold (A^u).[8] We normalize asset supplies so that both monies yield the same dividend, $d > 0$. Potential money supplies are $\bar{A}^g$ and $\bar{A}^u$ with $\bar{A} = \bar{A}^g + \bar{A}^u$. Mining efficiencies are given by λ^g and λ^u. We assume $\lambda^g \bar{A}^g \geq \lambda^u \bar{A}^u$, which means that early on it is easier to mine silver rather than gold. We are interested in whether both monies can coexist, the timing of production, and the effects of competing monies on the overall stock of means of payment.

The HJB equation for the value function of a buyer with a total portfolio of $a = a^g + a^u$ units of money expressed in the numéraire solves

$$\rho V_t^b(a) = \max_{\mathbf{a}_t^* \geq 0} \left\{ \rho \left(a - a_t^*\right) + \mathbf{r}_t \mathbf{a}_t^* + \alpha \chi_t \left[W_t^b \left(\mathbf{a}_t^*\right) - V_t^b \left(a_t^*\right) \right] + \dot{V}_t^b(a) \right\}, \tag{30}$$

where $\mathbf{a}_t^* = \left(a_t^{g*}, a_t^{u*}\right)$ is the portfolio of targeted money holdings and $\mathbf{r}_t^* = \left(r_t^{g*}, r_t^{u*}\right)$ is the vector of rates of return. The first term is the cost of adjusting the portfolio from a to the targeted level, a_t^*. The second term is the return on the portfolio of monies. The third term is the surplus from a pairwise meeting that occurs at Poisson rate $\alpha \chi_t$. The bargaining solution in pairwise meetings treats both monies as perfect substitutes.[9]

[8] The periodic symbols of silver and gold are Ag and Au respectively.

[9] There are trading mechanisms in pairwise meetings that treat different assets differently, thereby explaining differences in rates of return. See, e.g., Zhu and Wallace (2007) and Nosal and Rocheteau (2013).

Hence, only the total wealth, a_t^*, matters when a match is formed and not its composition between the two monies. Using the expression for W_t^b given by (8), the HJB equation can be rewritten as:

$$\rho V_t^b(a) = \max_{\mathbf{a}_t^* \geq 0} \left\{ \rho \left(a - a_t^*\right) + \mathbf{r}_t \mathbf{a}_t^* + \alpha \chi_t \theta \left\{ u \left[q(a_t^*) \right] - q(a_t^*) \right\} + \dot{V}_t^b(a) \right\}. \tag{31}$$

The first-order conditions, assuming interior solutions, give:

$$\rho - r_t^g = \rho - r_t^u = \alpha \chi_t \theta \left[\frac{u' \left[q(a_t^*) \right] - 1}{(1-\theta) u' \left[q(a_t^*) \right] + \theta} \right]. \tag{32}$$

Since the two monies are perfect substitutes as means of payment in pairwise meetings, if they are held, then they must have the same rate of return. This rate of return equality and the fact that both monies pay the same dividend guarantees that the two monies have the same price, ϕ_t.

The HJB equation for the value function of a seller is generalized as follows:

$$\rho V_t^s = \max_{(e_t^u, e_t^g) \in \{0,1\}^2} \left\{ \lambda^u (\bar{A}^u - A_t^u) e_t^u \phi_t + \lambda^g (\bar{A}^g - A_t^g) e_t^g \phi_t + \alpha (1 - e_t^u - e_t^g)(1-\theta) \left[u\left(q_t\right) - q_t \right] + \dot{V}_t^s \right\}, \tag{33}$$

and $e_t^u + e_t^g \leq 1$. A seller can either mine gold ($e_t^u = 1$), silver ($e_t^g = 1$), or be an active producer ($e_t^u = e_t^g = 0$). The occupation choice corresponds to the following maximization problem:

$$\max \left\{ \alpha(1-\theta) \left[u\left(q_t\right) - q_t \right], \lambda^g (\bar{A}^g - A_t^g) \phi_t, \lambda^u (\bar{A}^u - A_t^u) \phi_t \right\},$$

where A^g and A^u are the amounts of silver and gold in circulation with $A = A^g + A^u$. In the following $m^g \equiv \int_0^1 e_{i,t}^g di$ is the measure of silver miners, $m^u \equiv \int_0^1 e_{i,t}^u di$ is the measure of gold miners, and $m = m^g + m^u$.

Under the assumption $\lambda^g \bar{A}^g \geq \lambda^u \bar{A}^u$, when A^g and A^u are close to 0, then only silver is mined, $A_t = A_t^g$. In the regime where only silver is mined, the indifference condition between mining and production is

$$\alpha(1-\theta) \left[u\left(q_t\right) - q_t \right] = \lambda^g (\bar{A}^g - A_t) \phi_t, \tag{34}$$

and the law of motion of the total asset supply is

$$\dot{A} = \lambda^g (\bar{A}^g - A) m. \tag{35}$$

When the supply of silver is sufficiently large, agents might have incentives to mine gold as well. Whenever the two assets are mined, $\lambda^g (\bar{A}^g - A^g) = \lambda^u (\bar{A}^u - A^u)$, which implies

$$\bar{A}^g - A^g = \frac{\lambda^u}{\lambda^g + \lambda^u} (\bar{A} - A).$$

The fraction of unmined silver is a constant fraction of the total quantity of unmined assets. The indifference condition between production and mining can be rewritten as

$$\alpha(1-\theta)\left[u\left(q\right)-q\right]=\frac{\lambda^{g}\lambda^{u}}{\lambda^{g}+\lambda^{u}}(\bar{A}-A)\phi. \tag{36}$$

Note that this condition is identical to the one in the one-asset economy where the effective mining rate is $\lambda=\lambda^{g}\lambda^{u}/(\lambda^{g}+\lambda^{u})$. For instance, if $\lambda^{g}=\lambda^{u}=\lambda$ then $\bar{A}^{g}-A^{g}=\bar{A}^{u}-A^{u}=(\bar{A}-A)/2$ and the expected return from mining is $\lambda(\bar{A}-A)\phi/2$. Agents remain indifferent between mining silver or gold if $\lambda^{g}\dot{A}^{g}=\lambda^{u}\dot{A}^{u}$, which implies $\lambda^{g}m^{g}=\lambda^{u}m^{u}$, where m^{g} is the measure of silver miners, m^{u} is the measure of gold miners, and $m=m^{g}+m^{u}$. The growth of the total supply of assets, $\dot{A}=m^{g}\lambda^{g}(\bar{A}^{g}-A^{g})+m^{u}\lambda^{u}(\bar{A}^{u}-A^{u})$, is equal to

$$\dot{A}=\frac{\lambda^{u}\lambda^{g}}{\lambda^{g}+\lambda^{u}}m(\bar{A}-A). \tag{37}$$

The phase diagram corresponding to this dual asset economy is represented in Figure 6. The indifference condition, (34), corresponds to the upward-sloping purple curve labeled MG. The indifference condition, (36), corresponds to the red upward-sloping curve labeled $M2$. For an equilibrium starting from $A^{u}=A^{g}=0$, the relevant indifference condition is the frontier of the yellow area where $m=1$, i.e.,

$$\alpha(1-\theta)\left[u\left(q\right)-q\right]=\max\left\{\lambda^{g}(\bar{A}^{g}-A)\phi,\frac{\lambda^{u}\lambda^{g}}{\lambda^{g}+\lambda^{u}}(\bar{A}-A)\phi\right\}.$$

The two indifference loci, MG and $M2$, intersect for a positive ϕ, as illustrated in Figure 6, only if $\lambda^{g}\bar{A}^{g}>\lambda^{u}\bar{A}^{u}\left(\alpha+\lambda^{g}\right)/\alpha$. Otherwise, $M2$ is located to the right of MG. The level of assets at which agents transition from mining silver only to mining both gold and silver is $\hat{A}\equiv\left(\lambda^{g}\bar{A}^{g}-\lambda^{u}\bar{A}^{u}\right)/\lambda^{g}$.

Proposition 4 ***(Dual asset equilibrium)*** *Assume $A_{0}^{u}=A_{0}^{g}=0$. For all $\bar{A}^{u}>0$, there is a $\kappa_{0}>0$ independent of $\{\lambda^{u},\lambda^{g},\bar{A}^{u},\bar{A}^{g}\}$ such that if*

$$\frac{\lambda^{g}\bar{A}^{g}-\lambda^{u}\bar{A}^{u}}{\lambda^{g}\lambda^{u}\bar{A}^{u}}<\kappa_{0}, \tag{38}$$

then the long-run equilibrium features two competing assets as media of exchange. For all t such that $A_{t}<\hat{A}\equiv\left(\lambda^{g}\bar{A}^{g}-\lambda^{u}\bar{A}^{u}\right)/\lambda^{g}$, the economy has a single means of payment, $m_{t}^{u}=A_{t}^{u}=0$; For all t such that $A_{t}>\hat{A}$, the economy has two means of payment with

$$A_{t}^{u}=\frac{\lambda^{g}}{\lambda^{u}}\left(A_{t}^{g}-\hat{A}\right)>0 \tag{39}$$

$$m_{t}^{u}=\frac{\lambda^{g}}{\lambda^{g}+\lambda^{u}}m_{t}>0. \tag{40}$$

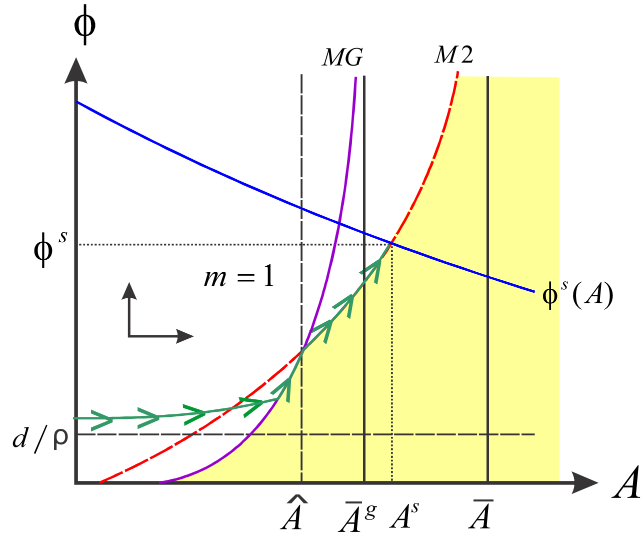

Figure 6: Phase diagram for a dual asset economy

According to (38), if the initial mining rates of the two assets are not too far apart (where mining rates are equal to the product of mining efficiency and asset supply), then the economy transitions from producing and using a single asset as medium of exchange to using two assets. The transition takes place when the supply of the first money (silver) reaches some threshold $\hat{A}$, at which point the mining rates of the two monies are equalized. According to (40) when the two monies coexist, then the measure of miners is allocated across the two monies according to the relative mining speeds, i.e., the money that is easier to mine receives more miners. From (40) the stock of both monies are proportional, where the coefficient of proportionality is the ratio of the mining speeds of the two monies.

The steady state with two monies can be reduced to a pair (ϕ^s, A^s) solution to (32) and (36), i.e.:

$$\frac{d}{\phi^s} + \alpha\theta \left[\frac{u'[q(\phi^s A^s)] - 1}{(1-\theta)u'[q(\phi^s A^s)] + \theta} \right] - \rho = 0 \tag{41}$$

$$\alpha(1-\theta)\left\{u[q(\phi^s A^s)] - q(\phi^s A^s)\right\} - \frac{\lambda^g \lambda^u}{\lambda^g + \lambda^u}(\bar{A} - A^s)\phi^s = 0. \tag{42}$$

Equation (41) corresponds to the downward-sloping blue curve in Figure 6. As the asset supply increases, the liquidity premium on assets decreases, and hence the asset price decreases. Equation (42) corresponds to the upward-sloping red curve, $M2$, in Figure 6. As the asset supply increases, the seller's surplus increases while the surplus of the miner decreases. In order to keep the indifference between the two occupations, the asset price must increase.

We can use this characterization of the steady state to study the effects of currency competition on prices and asset supply. An increase in the potential supply or mining efficiency of one of the two monies shifts the $M2$ curve downward, which leads to a fall of the common price of the two monies and an increase in the overall supply. Provided that $\phi^s A^s < \omega(q^*)$, then aggregate real balances and output in pairwise meetings increase. Hence, steady-state welfare increases. If $\phi^s A^s \geq \omega(q^*)$ or if the two monies are almost fiat, $d \approx 0$, then output is unaffected.

We can also use the phase diagram in Figure 6 to study the dynamics of asset prices following the introduction of a new money. Suppose that initially only the silver money exists. The steady state is at the intersection of the curves ϕ^s and MG. After the steady state has been reached, a new money, gold, is introduced. The price of money falls immediately to bring the economy on the $M2$ locus and the price of money increases afterwards to reach a new steady-state level that is lower. Hence, the introduction of new monies can generates fluctuations in the price of money.

4.2 Private and government monies

Now we investigate the competition between a fiat money supplied by the government and privately-produced fiat money. We ask under which conditions the two monies coexist and whether the government can choose monetary policy to prevent the production of the private money.

Suppose that there are two fiat monies: a privately produced money (labelled b) and a government-produced money (labelled g). The mining rate of the private money is $\lambda(\bar{A}_b - A_b)$ where $\bar{A}_b$ is the potential supplied and A_b is the amount produced up to t. In order to determinate the exchange rate, the two monies differ by their acceptabilities in different meetings (e.g., Lester et al., 2012). There is fraction γ_b of meetings where only privately-produced money is acceptable. For instance, the private money is accepted as means of payment for illegal transactions, e.g., on online black markets such as Silk Road, whereas government money is not.[10] By symmetry, there is a fraction γ_g of meetings where only government-produced money is acceptable, e.g., in transactions that involve the government.[11] In the remaining fraction, γ_2, both types of monies are acceptable.[12] We denote $\alpha_j \equiv \alpha\gamma_j$ for $j \in \{b, g, 2\}$.

[10] Foley et al. (2018) found that approximately one-quarter of Bitcoin users are involved in illegal activity and estimate that around $76 billion of illegal activity per year involves Bitcoin (46% of Bitcoin transactions), which is close to the scale of the US and European markets for illegal drugs.

[11] In the search-theoretic model of Aiyagari and Wallace (1997) and Li and Wright (1998) the government is a positive measure of agents who participate in the matching process with private agents and whose trading strategies, e.g., which objects they accept in payments, are part of the government policy.

[12] This description with three types of meetings where assets have different acceptabilities is analogous to the model with money and bonds of Rocheteau et al. (2018).

The HJB equation for the value function of a buyer becomes:

$$\rho V_t^b(a) = \max_{\mathbf{a}_t^* \geq 0} \left\{ \Upsilon_t + \rho\left(a - a_t^*\right) + \mathbf{r}_t \mathbf{a}_t^* + \sum_{j \in \{b,g,2\}} \alpha_j \chi_t \theta \left\{ u\left[q_j(\mathbf{a}_t^*)\right] - q_j(\mathbf{a}_t^*) \right\} + \dot{V}_t^b(a) \right\}, \tag{43}$$

where Υ_t is a lump-sum transfer to buyers corresponding to the growth of the government money, $\mathbf{a}_t^* = (a_t^{b*}, a_t^{g*})$ is the portfolio of two monies and $\mathbf{r}_t^* = (r_{b,t}^*, r_{g,t}^*)$ is the vector of rates of return. The probability that a randomly met seller is a producer is $\chi_t = 1 - m_t$ where m_t is the measure of sellers producing the private money. The novelty on the right side is the fourth term that distinguishes the surpluses in three types of matches. The output in match j, $q_j(\mathbf{a}_t^*)$, is a function of the portfolio of currencies and not simply its total value. According to the Kalai solution,

$$\omega\left[q_j(\mathbf{a}_t^*)\right] = \min\left\{ \mathbb{I}_{j \in \{b,2\}} a_t^{b*} + \mathbb{I}_{j \in \{g,2\}} a_t^{g*}, \omega(q^*) \right\},$$

where $\mathbb{I}_j$ are indicator functions equal to one if the currency is accepted in match j. The first-order conditions for the optimal portfolio of currencies generate the following Euler equations for $j = b, g$:

$$\rho - r_j = \alpha_j(1-m)\theta\left[\frac{u'\left(q_j\right) - 1}{(1-\theta)u'\left(q_j\right) + \theta}\right] + \alpha_2(1-m)\theta\left[\frac{u'\left(q_2\right) - 1}{(1-\theta)u'\left(q_2\right) + \theta}\right] \tag{44}$$

where q_j indicates output in a match of type $j \in \{b, g, 2\}$. The term on the left side, $\rho - r_j$, is the cost of holding money j. The terms on the right side represent the liquidity services that money j provides at the margin in different types of matches. The second term is common to both monies while the first term is specific to each money.

The HJB equation for the value function of a seller is now:

$$\rho V_t^s = \max_{e_t \in \{0,1\}} \left\{ \lambda(\bar{A}_b - A_{b,t}) e_t \phi_{b,t} + \alpha(1-e_t)(1-\theta) \sum_{j \in \{b,g,2\}} \left[u\left(q_{j,t}\right) - q_{j,t}\right] + \dot{V}_t^s \right\}. \tag{45}$$

The first term is the revenue of a miner while the second term is the expected surplus of a producer. The measure of miners is $m_t = \int_0^1 e_t^i di$. The indifference condition between occupations is

$$\lambda(\bar{A}_b - A_b)\phi_b = \sum_{j \in \{b,g,2\}} \alpha_j (1-\theta)\left[u\left(q_j\right) - q_j\right]. \tag{46}$$

As before, this condition gives a positive relationship between ϕ_b and A_b. The law of motion for A_b is

$$\dot{A}_b = \lambda(\bar{A}_b - A_b)m. \tag{47}$$

We assume that monetary policy aims at keeping q_g constant by varying r_g — a form of price level targeting. As a result, aggregate real balances supplied by the government, $\omega(q_g) = \phi_g A_g$, are constant. It follows that the quantity traded in type-b and type-2 matches solve:

$$\omega(q_2) = \min\{\omega(q_g) + \phi_b A_b, \omega(q^*)\} \quad (48)$$

$$\omega(q_b) = \phi_b A_b. \quad (49)$$

If the total market capitalization of the two monies (in terms of the numéraire) is larger than $\omega(q^*)$, then agents trade q^* in type-2 matches and spend only a fraction of their money holdings. Otherwise, they spend all their money, both private and public. An equilibrium is a list $(r_b, r_g, \phi_b, q_2, q_b, m, A_b)$ that solves (44), (48), (49), (46), (47) and $r_b = \dot{\phi}_b/\phi_b$.

We represent equilibria on the phase diagram in Figure 7. The indifference locus between occupations, (46), is represented by an upward-sloping red curve. The isocline for ϕ_b, from (44), with $m = 0$, is such that

$$\rho = \theta\left\{\alpha_b\left[\frac{u'(q_b) - 1}{(1-\theta)u'(q_b) + \theta}\right] + \alpha_2\left[\frac{u'(q_2) - 1}{(1-\theta)u'(q_2) + \theta}\right]\right\}, \quad (50)$$

where q_b and q_2 are given by (48)-(49). It is represented by a downward-sloping blue curve. The dynamics are qualitatively similar to the ones studied earlier. In particular, ϕ_b and A_b increase over time until they reach a steady state. Since there is a stock of government money, some agents might choose to become producers even when $A_b = 0$, as illustrated on the left panel of Figure 7. On the right panel, all agents are miners initially and a fraction start producing when the quantity of private money is sufficiently abundant. Note that when $m = 1$ the rate of return of the government money must be $r_g = \rho$ for agents to be willing to hoard it until it can serve as means of payment.

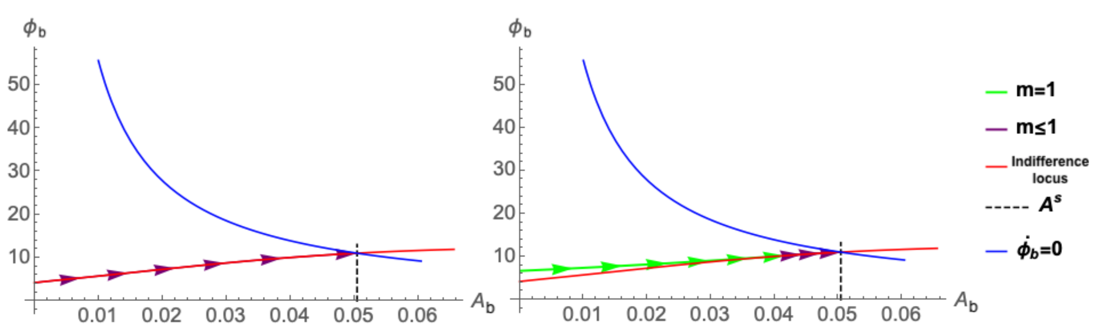

Figure 7: Two numerical examples of equilibrium. (Left) Agents are indifferent between producing and mining even when $A_b = 0$. (Right) Agents specialize in mining early on.

At the steady state $m = 0$ and $r_b = 0$, which pins down q_b and q_2. The indifference condition between occupations gives ϕ_b. We can now determine a condition under which the government can prevent the emergence of the private money.

Proposition 5 ***(Preventing the emergence of a private money)*** *There is no equilibrium with production of private money if*

$$\frac{\alpha_b \theta}{1-\theta} + \alpha_2 \theta \left[\frac{u'(q_g) - 1}{(1-\theta) u'(q_g) + \theta} \right] < \rho. \tag{51}$$

The proof based on (50) is straightforward and is therefore omitted. By raising q_g the government reduces the liquidity shortage, and hence incentives to produce the private money. However, if $\alpha_b \theta > \rho(1-\theta)$, even the Friedman rule ($q_g = q^*$) is not enough to eliminate the private money. The government must reduce the fraction of matches where the private money is the only means of payment.

5 Conclusion

This paper is motivated by the recent rise of private production of cryptocurrencies, and its similarity to the mining of commodity money such as gold coins. We propose and solved a continuous-time random-matching model for studying the production and pricing of divisible private monies, both in and out of steady state, with single or dual currencies. We consider time-consuming mining technologies that correspond to gold mining and crypto mining. The aggregate real balances and money supply can increase over time even when money is not providing any transactional services. We consider economies with two competing private monies, and derive conditions under which one or both of them are produced and used as media of exchange. Finally we study the competition between a fiat government money and a private money and derive conditions such that the government can use monetary policy to prevent the emergence of private money. Our model is flexible and can be extended to address various topics related to private monies, for example, the competition among currency designers and the role of miners of crypto-currencies to authenticate and validate transactions.

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Appendix: Omitted Proofs

Proof of Proposition 1. Part 1. Consider a steady state with abundant liquidity such that $q^s = q^*$ and $\phi^s = d/\rho$. Such a steady state requires that $\omega(q^*) \leq \phi^s A^s$. This equilibrium exists if the curve representing (19) is located below the curve representing (20) when $A = \rho\omega(q^*)/d$. Alternatively, the left side of (19) is greater than the right side when $\phi = d/\rho$ and $A = \rho\omega(q^*)/d$, i.e., (21) holds.

Suppose a steady state with abundant liquidity exists. From (18)

$$\frac{\dot{\phi}}{\phi} = \frac{\rho(\phi - \phi^s)}{\phi} - \alpha(1-m)\theta\left[\frac{u'(q_t) - 1}{(1-\theta)u'(q_t) + \theta}\right].$$

In the neighborhood of the steady state $q_t = q^*$ and the second term on the right side is equal to 0. The only solution to $\dot{\phi} = \rho(\phi - \phi^s)$ is such that $\phi_t = \phi^s$. Any other path violates $\lim_{t\to+\infty} e^{-\rho t}\phi_t = 0$ or $\phi_t \geq d/\rho$. Solving $\dot{\phi} = \rho(\phi - \phi^s)$ in backward time leads to $\phi_t = d/\rho$ for all t. Using that $\lambda(\bar{A} - A^s)\phi = \alpha(1-\theta)[u(q^*) - q^*]$ at the steady state, it follows that $\lambda(\bar{A} - A_t)\phi > \alpha(1-\theta)[u(q^*) - q^*]$ for all t such that $A_t < A^s$, and hence $m_t = 1$ for all t such that $A_t < A^s$. See the right panel of Figure 2.

Given $m_t = 1$, by (16) the money growth rate is $\dot{A} = \lambda(\bar{A} - A)$. One can solve this ODE and the rest of Part 1 of Proposition 1 follows immediately.

Part 2. If (21) does not hold then the unique steady state features $q^s < q^*$ and $\phi^s > d/\rho$. By (16) and (18) the slope of the trajectory in the (A, ϕ) space is

$$\frac{\partial\phi}{\partial A} = \frac{\dot{\phi}}{\dot{A}} = \frac{\frac{1}{m}\left[\rho\phi - d - \left(\frac{\alpha\theta[u'(q)-1]}{(1-\theta)u'(q)+\theta}\right)\phi\right] + \frac{\alpha\theta[u'(q)-1]}{(1-\theta)u'(q)+\theta}\phi}{\lambda(\bar{A} - A)}. \tag{52}$$

The square bracketed term is strictly negative as $q < q^s$ and $\phi < \phi^s$ and by the definition of the steady state. Hence the right side rises in m.

Next we compute the slope $\partial\phi/\partial A$ of the indifference locus. Since q_t increases over time and $q^s < q^*$, the bargaining solution implies

$$\omega(q) = (1-\theta)u(q) + \theta q = \phi A. \tag{53}$$

By the implicit function theorem $\partial q/\partial(\phi A) = 1/\omega'(q)$ and thus $S'(\phi A) = [u'(q) - 1]/\omega'(q)$. By (19), (53) and the definition of S, the slope of the indifference locus can be written as

$$\frac{\partial\phi/\phi}{\partial A/A} = \frac{\alpha(1-\theta)\frac{[u'(q)-1]}{\omega'(q)} + \lambda}{\alpha(1-\theta)\left[1 - \frac{\omega(q)[u'(q)-1]}{\omega'(q)[u(q)-q]}\right]\frac{[u(q)-q]}{\omega(q)}}. \tag{54}$$

Let $m_I(q,\phi)$ be the measure of miners implied by the indifference locus. To solve for $m_I(q,\phi)$ replace $\partial\phi/\partial A$ in the above equation by (52). Then by (19) and (53)

$$\left\{1-\frac{\omega(q)[u'(q)-1]}{\omega'(q)[u(q)-q]}\right\}\left\{\frac{\rho-d/\phi-\frac{\alpha\theta[u'(q)-1]}{\omega'(q)}}{m_I(q,\phi)}+\frac{\alpha\theta[u'(q)-1]}{\omega'(q)}\right\}=\alpha(1-\theta)\frac{[u'(q)-1]}{\omega'(q)}+\lambda \qquad (55)$$

Suppose the dynamical system starts from the steady state and goes backward in time. When the trajectory lies strictly above the indifference locus, then clearly $m_t=1$. When it lies exactly on the indifference locus, it is in the regime with $m_t\in[0,1)$ if $m_I(q_t,\phi_t)<1$ and otherwise $m_t=1$. For if $m_I(q_t,\phi_t)<1$ then the trajectory cannot have $m_t=1$, as the backward-time trajectory would go beneath the indifference locus which is impossible (recall that (52) rises in m). If $m_I(q_t,\phi_t)\geq 1$, then the trajectory clearly cannot be in the regime with $m_t\in[0,1)$.

Next we argue the equilibrium leading to (A^s,ϕ^s) is unique. Consider the backward time trajectory starting from (A^s,ϕ^s). If it lies on the indifference locus (i.e. in the regime $m\in[0,1]$), then clearly it is unique. If it switches regime to $m=1$ at some $(\hat{A},\hat{\phi})$, then the trajectory is characterized by (52). By Theorem 58.5 in Tennenbaum and Pollard (1985), there is a unique solution when running (52) in backward time provided that the right side of (52) is Lipschitz continuous, namely if the right side of (52) is $f(A,\phi)$, then we require a real constant $K>0$ such that

$$|f(A,\phi')-f(A,\phi'')|\leq K|\phi'-\phi''|$$

for every $A\in[0,\hat{A}]$ and $\phi',\phi''\in[0,\hat{\phi}]$. For any $A\in[0,\hat{A}]$, one can check that the slope of f with respect to ϕ is bounded provided that u'' is bounded. Therefore, f is Lipschitz continuous and the equilibrium trajectory is unique.

Finally the economy has $m_t=1$ near the steady state if and only if $m_I(q_t,\phi_t)\geq 1$ near the steady state. Equivalently near the steady state the slope of the trajectory when $m=1$,

$$\frac{\partial\phi/\phi}{\partial A/A}=\alpha\theta\frac{S(A\phi)}{\lambda(\bar{A}-A)\phi}\nu(A\phi),$$

is larger than the slope of the locus where agents are indifferent between occupations, i.e.,

$$\frac{\partial\phi/\phi}{\partial A/A}=\frac{\nu(A\phi)+A/(\bar{A}-A)}{1-\nu(A\phi)}.$$

By (19) this happens when (22) does not hold near the steady state. Otherwise, the equilibrium features $m_t<1$ in the neighborhood of the steady state. Lemma 1 proves the last claim concerning the number of regime switches.

Part 3. From (20) if $d = 0$ then q^s solves

$$\rho = \alpha\theta \left\{ \frac{u'(q) - 1}{(1-\theta)u'(q) + \theta} \right\}.$$

A unique solution exists if $\rho < \alpha\theta/(1-\theta)$. The characterization of the unique equilibrium leading to the steady state is similar to Part 2. The characterization of the continuum of boom-burst equilibria is identical to that in the proof of Proposition 3 in Choi and Rocheteau (2019). ■

Lemma 1 *If liquidity is scarce as described by Part 2 of Proposition 1 and the elasticity $u'(q)q/u(q)$ falls in q, then the equilibrium trajectory at most has one regime switch, from $m = 1$ to $m < 1$.*

Proof. We first show $m_I(q,\phi)$ defined by (55) fall as q and ϕ increase, provided that $m_I(q,\phi) < 1$. The right side of (55) falls strictly in q by the concavity of u. The left side of (55) rises in ϕ because $d \geq 0$. The first term in the second large braces on the left side of (55) is strictly negative when $q < q^s$ and $\phi < \phi^s$ by the definition of the steady state. Therefore, the left side rises strictly in m_I. The second large bracket in the left side of (55) rises in q by the concavity of u and $m_I(q,\phi) < 1$. If the elasticity of u falls in q then the expression in the first large bracket in the left side also rises in q, and thus the left side rises in q. Altogether as q and ϕ increase, $m_I(q,\phi)$ must fall strictly to balance the equation.

Suppose the dynamical system starts from the steady state and goes backward in time. The backward time dynamical system has at most one regime switch from $m_t < 1$ to $m_t = 1$. It is because (i) the trajectory is in the regime $m_t < 1$ iff $m_I(q_t,\phi_t) < 1$ as explained in the proof of Part 2 of Proposition 1, (ii) q_t and ϕ_t fall as the trajectory go backward in time and (iii) $m_I(q,\phi)$ falls as q and ϕ increase.

Since $m_t = 1$ when A is sufficiently small, as t increases from 0 the equilibrium either stays in the regime with $m_t = 1$ or has exactly one regime switch from $m_t = 1$ to $m_t < 1$. ■

Proof of Proposition 4. As argue in the main text, there is production of both gold and silver along the equilibrium path if $\hat{A} < A^s$. Graphically, this condition is satisfied if the intersection of the loci MG and $M2$, $(\hat{A},\hat{\phi})$, is located below the locus $\phi^s(A)$. See Figure 6. Let $\hat{Z} \equiv \hat{\phi}\hat{A}$ denote the aggregate liquidity at the intersection of MG and $M2$. At the intersection $A^u = 0$ and the agents are indifferent between producing or mining gold, therefore $\alpha(1-\theta)S(\hat{Z}) = \hat{Z}\lambda^u\bar{A}^u/\hat{A}$. By the definition of $\hat{A}$, $\hat{Z}$ is the largest solution of

$$\alpha(1-\theta)S(\hat{Z}) = \hat{Z}\frac{\lambda^g\lambda^u\bar{A}^u}{\lambda^g\bar{A}^g - \lambda^u\bar{A}^u}.$$

This solution is strictly positive if $\lambda^g \bar{A}^g > \lambda^u \bar{A}^u (\alpha + \lambda^g) / \alpha$ and $\hat{Z} = 0$ otherwise. Moreover, $\hat{Z}$ increases with $(\lambda^g \bar{A}^g - \lambda^u \bar{A}^u)/\lambda^g \lambda^u \bar{A}^u$. By (20) the condition $\hat{A} < A^s$ is equivalent to:

$$\rho < \frac{d}{\hat{\phi}} + \alpha\theta S'(\hat{Z}) \iff \rho < \frac{d\lambda^u \bar{A}^u}{\alpha(1-\theta)S(\hat{Z})} + \alpha\theta S'(\hat{Z}).$$

The right side is decreasing in $\hat{Z}$ and it tends to infinity as $\hat{Z}$ goes to 0. Hence, this condition is satisfied provided that $(\lambda^g \bar{A}^g - \lambda^u \bar{A}^u)/\lambda^g \lambda^u \bar{A}^u$ is sufficiently low, i.e.,

$$\frac{\lambda^g \bar{A}^g - \lambda^u \bar{A}^u}{\lambda^g \lambda^u \bar{A}^u} < \kappa_0$$

for $\kappa_0 > 0$. The threshold κ_0 is endogenous but by construction it does not depend on λ^g and $\bar{A}^g$. This implies that, fixing all parameters except $\bar{A}^g$, the inequality is satisfied as $\bar{A}^g \to \lambda^u \bar{A}^u / \lambda^g$ from above. Similarly the inequality is satisfied when λ^g is sufficiently small. The construction of the equilibrium going backward from the steady state is as described in earlier proposition and is therefore omitted. ■

Online Appendix for "Money Mining and Price Dynamics"

February 2020

General matching function

In the main text we assume an uniform random matching function, suppose now only buyers (money holders) and producers participate in the matching process according to a constant returns to scale matching function. We generalize the matching function in order to obtain interior solutions for the occupation choice and we describe how the matching technology can affect price dynamics. Suppose that each agent receives an opportunity to consume at Poisson arrival rate $\alpha(\tau)$ where τ is the measure of producers per consumer, i.e., the tightness of the goods market. Because the measure of consumers is one while the measure of producers is $1-m$, tightness is simply $\tau = 1-m$. Each of the $1-m$ producers is matched with a consumer at Poisson arrival rate $\alpha(1-m)/(1-m)$. The matching function used so far is $\alpha(\tau) = \tau$. In order to guarantee that $m_t < 1$ throughout the equilibrium path, we impose $\alpha'(0) = +\infty$. Hence, the matching rate of a producer as m approaches 1 is $\lim_{m \nearrow 1} \alpha(1-m)/(1-m) = \alpha'(0) = +\infty$. Provided that $\theta < 1$, it is always optimal for some agents to choose the production sector over mining.

The measure of agents in the mining sector solves:

$$\frac{\alpha(1-m)}{1-m} \geq \frac{\lambda\left(\bar{A}-A\right)\phi}{(1-\theta)S(\phi A)}, \quad "=" \text{ if } m>0,$$

where $S(\phi A) = u(q) - q$ with $\omega(q) = \min\{\omega(q^*), \phi A\}$. We denote $m(\phi, A)$ the solution to this equation. For all (ϕ, A) such that $\lambda\left(\bar{A}-A\right)\phi > \alpha(1)(1-\theta)S(\phi A)$, $m(\phi, A) > 0$ is an increasing function of ϕ and a decreasing function of A. Otherwise,

$$m(\phi, A) = 0 \;\text{ if }\; \alpha(1)(1-\theta)S(\phi A) \geq \lambda\left(\bar{A}-A\right)\phi.$$

The money supply evolves according to

$$\dot{A} = \lambda m(\phi, A)\left(\bar{A}-A\right).$$

Hence, $\dot{A} = 0$ if

$$\frac{\alpha(1)(1-\theta)}{\lambda} \geq \frac{\left(\bar{A}-A\right)\phi}{S(\phi A)}.$$

The frontier of this region in the (A,ϕ) space is upward-sloping, it has $A=\bar{A}$ as a vertical asymptote, and it goes through the origin.

The ODE for the value of the asset is

$$\rho\phi = d + \alpha\left[1-m(\phi,A)\right]\theta S'(\phi A)\phi + \dot{\phi},$$

where $q(\phi A)$ solves $\omega(q)=\min\{\omega(q^*),\phi A\}$. The ϕ-isocline is the locus of the pairs, (A,ϕ), such that $\dot{\phi}=0$, i.e.,

$$\rho = \frac{d}{\phi} + \alpha\left[1-m(\phi,A)\right]\theta S'(\phi A).$$

The ϕ-isocline needs not be monotone. To see this, note that when A is small, $m(\phi,A)\approx 1$, which gives a positive relationship between ϕ and A. When $\phi A \approx \omega(q^*)$, then the isocline is downward-sloping.

Suppose the stationary equilibrium with $m=0$ is such that $\phi A \geq \omega(q^*)$. The asset is priced at its fundamental value, $\phi = d/\rho$, and the asset supply is

$$A = \frac{\lambda d\bar{A} - \rho\alpha(1)(1-\theta)\left[u(q^*)-q^*\right]}{\lambda d}.$$

This equilibrium exists if

$$\lambda d\bar{A} \geq \lambda\rho\omega(q^*) + \rho\alpha(1)(1-\theta)\left[u(q^*)-q^*\right].$$

In the neighborhood the ϕ-isocline is horizontal and when ϕA is slightly less than $\omega(q^*)$ is is downward-sloping. It implies that the price of the asset is larger than the fundamental value initially and it reaches the fundamental value when the asset supply becomes sufficiently abundant.

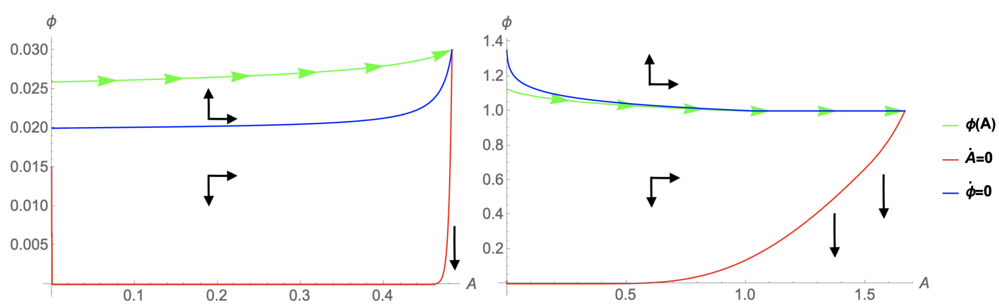

Figure 8: Two numerical examples: (Left) Scarce liquidity. (Right) Abundant liquidity.

In Figure 8 we represent the phase diagrams for two numerical examples. The blue curve is the ϕ-isocline such that $\dot{\phi}=0$. The red curve is the A-isocline such that $\dot{A}=0$. The green curve with arrows corresponds to the saddle path leading to the steady state. In the left panel, liquidity is scarce at the steady state. The asset price, ϕ_t, is strictly above its fundamental value, d/ρ, and it rises over time. These dynamics are similar to the ones described in part 2 of Proposition 1 except that there is production of the consumption good throughout the equilibrium path. In the right panel, liquidity is abundant at the steady state. Now the asset price, ϕ_t, falls over time and it converges to its fundamental value. These dynamics are new and illustrate how the matching technology matters for the time path of asset prices.