Optimal Sequential Search

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Abstract

We introduce a simple new model of sequential search among finitely many options that fits many economic applications. Each payoff is the sum of a random "known factor" and a "hidden factor", learned at cost. Weitzman (1979) solved the *ex post* Pandora's box problem, given known factors. Ours is the *ex ante model* for estimation, unconditional on known factors, and so resolves major *selection effects*.

1. Search intensifies over time, as one increasingly exercises the current option, recalls a prior one, or quits. If one recalls, earlier options are recalled more often.

2. We solve a long open question in all search models: which stochastic changes lead one to search longer? Answer: more *dispersed* payoffs.

3. The stationary search model poorly approximates search with many options: If the known factor density lacks a thin tail (eg. exponential), the recall chance is boundedly positive with vastly many options.

4. Search lasts longer with more options. Hence, if low search frictions increase worker applicant pools of firms, vacancy duration increases.

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1 Introduction

Sequential search analyzes the classic "now or later" problem in economics. It is also practically very important, as the search for goods or jobs or ideas or people occupies much of our days. Yet rarely does it transpire in the zero information vacuum of the standard search model (McCall, 1970) with an unbounded number of identical options. Those hunting for purchases invariably know which stores are closer; firms hiring workers can easily observe their college of origin; and those seeking romantic partners quickly perceive looks or location. Finally, nowadays one oft searches through smartly pre-sorted lists in more than 100 billion monthly Google queries, Netflix movie quests, Monster job hunts, or Amazon searches. But even if the stationary benchmark does not resemble actual search environments, its predictions may still be pretty good. We argue not.

We propose and solve a dynamic search model with finitely many heterogenous risky *inside options*. Payoffs are the sum of a *known* factor and another *hidden* until a look-see cost is paid. A searcher, called Sam, can *exercise* just one option, including a quit payoff. Weitzman (1979) solved the "Pandora's Box" problem with fixed known factors: Sam optimally explores options in order of known factors, until either quitting, recalling an already explored option, or exercising the current option (*striking*). We instead explore Sam's behavior as perceived by an outside observer for whom known factors are random.

Sam is initially forward-looking, torn only between striking and *passing*. But since the best fallback option improves over time and expected future prospects dim, search concludes at some *recall moment*: Sam's choice margin is then strike, recall, or quit (Lemma 3). While Sam's search ends not with a whimper but with a bang, this recall moment is random from our outside observer's perspective. For a wide reach of our theory, we just assume that factors have log-concave distributions (a widely met condition): This implies that search intensifies over time: conditional chances of quitting, recalling, and exercising all increase period by period (Theorem 1), and when recall occurs, earlier options are recalled more often (Theorem 2). Logconcavity ultimately ensures that the direct effect of a higher known factor (stop now) dominates the signaling value of it (stopping later is more profitable), and so the striking chance should rise in the known factor. These are the first robust and testable results on recall chances in search theory.

We next turn to an important and yet long open question in search: Which general prize distribution changes increase search duration? For instance, a more talented worker or more prized match partner each has a richer opportunity set. Are they tempted to search longer for jobs or mates? It is unclear, as Sam responds to first order stochastic dominance and mean preserving spreads (MPS) of the prizes by more aggressive search (Mortensen, 1987). But the higher optimal reservation threshold is offset by increased probability weight in all upper tails. Search duration rises only if the substitution effect of a higher reservation prize dominates the direct effect of better options.

We solve this puzzle with the *dispersion* stochastic order. A probability distribution rises in this order if the gap between all quantile pairs increases. Fixing the mean, this order everywhere "stretches" the distribution, and is stronger than MPS. *For the benchmark undiscounted stationary search model, duration is higher with a more dispersed prize distribution* (Theorem 3).[1] We argue this by rewriting the search Bellman equation as "search cost equals the expected survivor probability, with respect to the quantile distribution". Finally, this expected marginal benefit of another search rises because the *quantile function* (inverse cdf) steepens in the dispersive order. We also directly argue this major result using intuitive economics — the "substitution effect" of more aggressive search dominates the "income effect" of more high prizes with more dispersion.

The dispersion order has two side benefits: First, it is practically measurable for empirical distributions, and second, a single parameter adjusts dispersion of many common distributions — for instance, the scale parameter of a Gamma distribution (see Table 1).

The increasing dispersion-duration link transfers to the hidden component in our two factor model: search duration is higher for a more dispersed hidden factor (Theorem 4). The logic is more subtle in this nonstationary setting; we argue instead that Sam passes to the next option more often with increased dispersion. Next, dispersion of the known factor has the opposite effect: Sam is less willing to continue searching (and also more willing to initiate search) with a more disperse known factor (Theorem 5). This is intuitive since increasing dispersion of the known factor corresponds to a better informed Sam. Alternatively, the continuation value falls faster with more dispersion in the known factor; this diminishes Sam's future prospects faster, and thereby hastens the arrival of the recall moment. Search duration is greatest when options are most ex ante similar.

We finally consider how the number of options impacts search behavior. For instance, web matching markets, like upstart rivals to match.com, boast of their pool of potential mates. But does a larger number of rank-ordered options accelerate matching? Notably, we find that approaching the benchmark infinite-option search model (McCall, 1970) poorly approximates behavior with a vast number of inside options. For while the conditional chance of recalling a prior option falls in the number of options, it only

[1]But for stationary discounted wage search, duration rises in dispersion for high enough search costs, and otherwise falls. See Appendix A. Less wage compression should induce longer unemployment spells.

vanishes in the limit if the known factor distribution has a "thin tail" (Theorems 6 and 7). Otherwise, without a thin tail, the chance of recall never vanishes because top order statistics of known factors have boundedly positive gaps. For instance, the top order statistics for the exponential distribution have constant gaps, irrespective of the total number of options. Table 1 also flags which distribution have thin tails: Gaussian or uniform. For instance, when Netflix adds more movies, users benefit, but their video search duration increases; further, if the video known component does not have a thin tail, a searcher has a boundedly positive chance of recalling a prior video.

Our model is designed for estimation — since consumer preferences and information about products (i.e. our known factors) searched are unobservable to econometricians. Our predictions are driven by selection effects that so far have not been well-captured in search theory. For his sampling payoff distributions are themselves random objects in our model. Sam's willingness to explore the next inside option signals more promising top known factors. For example, the selection effect logic says that hitting a later stage offers more damning evidence of Sam's fallback options, and makes continued searching more likely. In the event of recall, this evidence is more dire for the earlier options, which have been passed over more often; this suggests that they are less likely to be recalled. These two selection intuitions run counter to Theorems 1 and 2, respectively.

Our paper greatly enriches the dynamic programming analysis for stationary search. For instance, the value function in wage search is a constant function of the current wage, until the reservation threshold; after that, it coincides with the 45^o diagonal. Notice that the slope in each case indicates the acceptance chance — initially 0, and then 1. For Sam accepts all wages over a reservation wage. In our nonstationary world, the value plot is increasing and convex in the fallback prize, and the increasing slope is the probability of eventually exercising the fallback (Figure 2). Another important contrast with the standard search model is that the reservation prize exceeds the continuation value. For the fallback option offers Sam insurance, and encourages him to search more aggressively.

Literature. Our paper enhances the pure theory of sequential search. Economists often cite McCall (1970) as the first sequential search paper.[2] In fact, we name our searcher Sam in honor of Karlin (1962), who solved a sequential search problem much earlier — in a harder nonstationary finite horizon problem. We build on Weitzman's influential 1979 "Pandora's Box" model, the last major innovation in sequential search. His search model posited finitely many known sampling distributions with look-see costs,

[2]"For some reason there was a ten year lag between" Stigler's pioneering (1961) "work and subsequent variations that comprise the vast literature beginning in 1970" (Lippman and Mccall, 1976).

and its optimal policy was an index rule. Unlike stationary search, this policy sometimes recalls a previously explored option. While both elegant and conceptually innovative, Weitzman's model offered few new behavioral predictions, since the payoff distributions of the boxes was completely arbitrary.[3] Our model with random additive known factors yields a large random class of Weitzman models with many predictions.

We assumed that Sam is apprised of all known factors. But Sam behaves identically if, when deciding, he only foresees the very next known factor, but not later ones. As such, ours is a model of search with learning, and in Rosenfield and Shapiro's seminal 1981 paper on this topic, a reservation price rule need not be optimal: For one may accept some prices, but reject lower ones, having been encouraged to ambitiously continue searching. Our rank-ordered known factors precludes this pathological possibility.[4]

Ganuza and Penalva (2010) use the dispersive order to study information disclosure in auctions.[5] In a paper on risk notions, Chateauneuf, Cohen, and Meilijson (2004) include an example of stationary search with no discounting, in which duration rises if the reward distribution grows *location independent riskier* (Jewitt, 1989).

Our paper opens the door to formal analyses of age-old behavioral topics related to the curse of choice (Chernev, Bockenholt, and Goodman, 2015) — as Toffler (1970) long ago made famous with the concept of "overchoice". This phenomenon describes the despair with more options, as they solicit so much more investigative look-see effort. This notion only makes sense in a search model with ex ante different options, and thus our model is a formal analysis of optimal behavior in this world. This observation of decision fatigue reflects an unmodeled ex ante stage in which Sam rank order the known factors. A smart behavioral model of this procedure is an intriguing open problem.

The two factor search model is described in §2, and Sam's optimal behavior is quickly derived in §3. The paper then shifts perspective to an observer unaware of known factors. In §4, we flesh out the growing intensity of the search ending activities — striking, recall, and quitting. In §5, we study how prize dispersion impacts search duration. And in §6, we underscore the importance of our analysis, by questioning whether the stationary search benchmark model corresponds to the limit with infinitely many options. An appendix addresses discounted search, and contains the longer or more technical proofs.

[3] Olszewski and Weber (2015) find a more general index rule; Doval (2018) lets Sam freely exercise unexplored options. Sam can explore old options or find new ones in Fershtman and Pavan (2019).

[4] Recent work on optimal sequential search with learning includes Austen-Smith and Martinelli (2018); Gossner, Steiner, and Stewart (2018); Ke and Villas-Boas (2019).

[5] Zhou (2017) and Choi, Dai, and Kim (2018) use it to study pricing in discrete choice models; the latter is an equilibrium version of our search model where sellers post prices and buyers search.

2 A Two Factor Model of Search

A decision maker Sam must make a single choice from $N < \infty$ *inside options*, each denoted $(\mathcal{X}, \mathcal{Z})$, and one *outside option*; the latter has a known payoff $u \in \mathbb{R} \cup \{-\infty\}$. The inside option has random payoff $\mathcal{X} + \mathcal{Z}$, where $\mathcal{X}$ is the *known factor* and $\mathcal{Z}$ the *hidden factor*. Their respective densities g and h are log-concave, and thus so are their cdf's G and H.[6] Each has full support either on $\mathbb{R}$, or an interval subset of $\mathbb{R}$ (Table 1).

The modeler faces *prospective uncertainty*: The known factors $\mathcal{X}$ and hidden factors $\mathcal{Z}$ are each independent random variables to him. On the other hand, Sam first learns all N realized known factors $\mathcal{X} = \chi$ before search. For him, *ranked options* $(\chi, \mathcal{Z})$ matter.

Sam faces a sequential search exercise, and seeks to maximize his expected payoff. While searching at stages $n = 0, 1, \ldots, N$, Sam may *explore* any inside option: He pays the "look-see" or *search cost* $c > 0$ to learns its realized hidden factor $\mathcal{Z} = z$. He may then either (*i*) *strike*, by *exercising* the current option, consuming its payoff and stopping search; or (*ii*) *pass*, by exploring a new inside option next stage; or (*iii*) *recall*, by exercising a previously passed option, or (*iv*) *quit*, by exercising the outside option. Sam *participates* if he explores any inside option; otherwise, search Sam quits at $n = 0$.

If Sam exercises an option with payoff w at stage $n \in \{0, 1, \ldots, N\}$, his final payoff is the value of the exercised option, less the accumulated search costs, or $w - nc$. Search with no outside option is the special case with quit payoff $u = -\infty$. The *stationary search model* is the special case with a constant known factor $\mathcal{X}$ (i.e. point mass G).

This two-factor search model captures two broad search settings. First, one often has prior information over the options one faces — e.g. firms see college pedigrees of job applicants. Sam is endowed with an unbiased signal $\mathcal{X}$ of W from a quick synopsis, and a costly look-see resolves all remaining uncertainty $\mathcal{Z} = W - \mathcal{X}$ (Choi and Smith, 2016).

Second, the web's greatest advantage is robotically mediated search. After entering a query for a Google search or jobs (Monster) or mates (Match.com) or goods (Amazon) or movies (Netflix), Sam's query is translated into a score $\mathcal{X}$ (e.g. 95% Netflix "match score"). These scores also reflect knowledge of Sam's preferences. Absent scores, brief synopses are usually offered, which Sam projects to a known factor $\mathcal{X}$.

We have overstated Sam's informational needs, as he does not need the entire score list for his dynamic optimization: Sam needs only know the next known factor (see Lemma 1). Exploring the link incurs a clicking cost $c > 0$, and reveals the hidden factor.

[6] Log concavity ensures monotone means of truncated distributions (Heckman and Honore, 1990) — essential here. If g is log-concave, so is $1 - G$, by Prekopa's Theorem, and $g/[1 - G]$ is non-decreasing.

3 Optimal Stopping Characterization and Behavior

We first consider two extreme cases. A constant known factor $\mathcal{X} \equiv \chi > u$ captures finite horizon search from a fixed distribution. In this case, Sam employs a constant cutoff, and only recalls if he hits the last period.[7] With a degenerate hidden factor $\mathcal{Z} \equiv 0$, Sam perfectly sorts options, stopping at the first; once more, Sam never recalls. But in our model with non-degenerate random known and hidden factors $\mathcal{X}$ and $\mathcal{Z}$, Sam confronts a nontrivial nonstationary search problem, and always recalls with positive probability.

By Lemma 1 below, Sam should explore options in the rank order of the realized known factors: $\chi_1 \geq \chi_2 \geq \cdots \geq \chi_N$. (A search engine formally preorders options in this way, while a quick perusal of options yields this rank order.) Let F_n be the distribution of the random payoff $W_n = \chi_n + \mathcal{Z}_n$ for the option with known factor $\mathcal{X}_n = \chi_n$. Its realized payoff is $w_n = \chi_n + z_n$. Since $\chi_n \geq \chi_{n+1}$, the payoff distributions $F_n(w) = H(w - \chi_n)$ fall in n in the first-order stochastic dominance sense. Next, as in Weitzman (1979), implicitly define n *reservation prizes* $\{\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_N\}$ for entering stage n ("opening that box"):[8]

$$\bar{w}_n = -c + \bar{w}_n F_n(\bar{w}_n) + \textstyle\int_{\bar{w}_n}^{\infty} w dF_n(w). \tag{1}$$

Integration by parts yields $c = \int_{\bar{w}_n}^{\infty} 1 - F_n(z) dz$, as is standard in search theory. Since F_n stochastically falls in n, so too do the reservation prizes, namely, (★): $\bar{w}_1 \geq \cdots \geq \bar{w}_N$. By Weitzman (1979), Sam explores options in this order of the known factors.

Given inside option payoffs $w_1, w_2, \ldots, w_N$, the dynamic programming state variable is the *fallback*: initially $\Omega_0 = u$ and then $\Omega_n = \max(u, w_1, \ldots, w_n)$ for stages $n = 1, \ldots, N$.

Lemma 1 (Optimal Search) *Sam explores new options in the falling order (★) of reservation prizes. In stage n, he stops searching when $\Omega_n \geq \bar{w}_{n+1}$. Specifically, he strikes if $w_n \geq \max\{\bar{w}_{n+1}, \Omega_{n-1}\}$, and recalls any fallback if $\Omega_{n-1} \geq \max\{\bar{w}_{n+1}, w_n\}$.*[9]

Intuitively, Sam strikes if the present beats the past and the future, or $w_n \geq \max(\bar{w}_{n+1}, \Omega_{n-1})$. Sam quits / recalls if the past beats the present and future, or $\Omega_{n-1} \geq \max(\bar{w}_{n+1}, w_n)$. Sam passes if the future beats the past and present, or $\bar{w}_{n+1} > \Omega_n$. This nontrivial triple choice captures a richness of this setting over stationary search.

[7] The cutoff is the Weitzman index of each option (derived in (1)). His index formula coincide with the reservation wage formula in McCall's infinite horizon wage search model, embellished with recall.

[8] Log-concavity of H ensures finite moments, and thus $\bar{w}_n < \infty$ for all n

[9] The stopping event $\Omega_n \geq \bar{w}_{n+1}$ includes striking or recall: $w_n \geq \max\{\bar{w}_{n+1}, \Omega_{n-1}\}$ or $\Omega_{n-1} \geq \max\{\bar{w}_{n+1}, w_n\}$ is $\max\{w_n, \Omega_{n+1}\} \geq \bar{w}_{n+1}$, and $\max\{w_n, \Omega_{n+1}\} \equiv \Omega_n$, by definition. When indifferent, we assume Sam strikes. This choice is moot, since this has zero chance, since densities g and h exist.

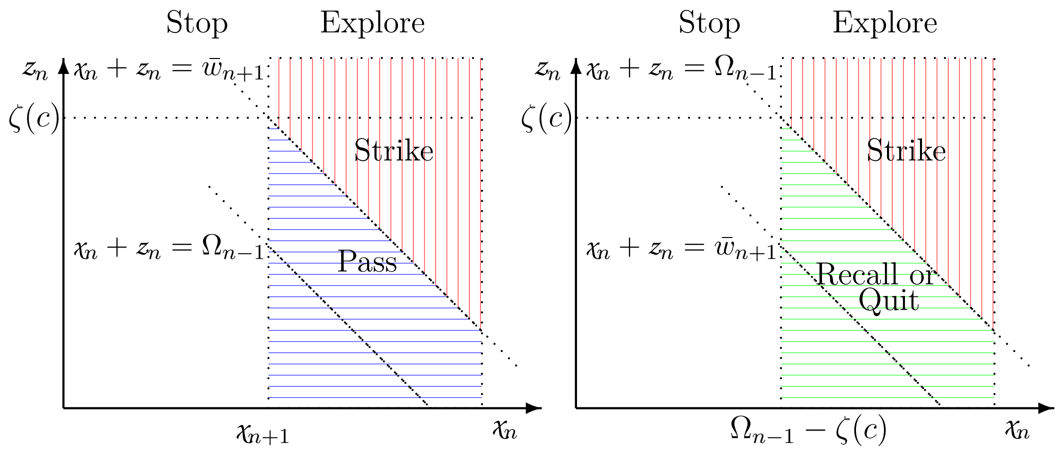

Figure 1: **The Phase Transition in Search.** We plot Sam's optimal behavior given the known and idiosyncratic factors, $\varkappa$ and z. Early on, when $\bar{w}_{n+1} > \Omega_{n-1}$, Sam always decides between strike and pass (left). But we eventually transition to $\Omega_{n-1} \geq \bar{w}_{n+1}$, whereupon Sam's decision margin shifts to strike or recall / quit (right).

Let the search *optionality value* $\zeta(c)$ be the reservation wage in stationary wage search with a zero known factor. As is well-known, this solves the discrete first order condition (FOC):

$$c = \int_{\zeta(c)}^{\infty} [1 - H(z)]dz. \tag{2}$$

In our two factor model, $\zeta(c)$ captures the upside benefits of the random hidden factor $\mathcal{Z}$.

Lemma 2 (Reservation Prizes) *In stage $n-1$, Sam accepts any $w \geq \bar{w}_n = \varkappa_n + \zeta(c)$.*

This expression follows from integrating the tail integral (1) by parts,[10] using (2):

$$c = \int_{\bar{w}_n}^{\infty} [1 - F_n(w)]dw = \int_{\bar{w}_n}^{\infty} [1 - H(w - \varkappa_n)]dw = \int_{\bar{w}_n - \varkappa_n}^{\infty} [1 - H(z)]dz.$$

The fallback Ω_n improves in n, while the reservation value $\bar{w}_n$ falls in n, by Lemma 1. Sam's future is initially brighter than his past, $\bar{w}_{n+1} \geq \Omega_{n-1}$, and he either passes or strikes. But the $\varkappa_n + z_n = \bar{w}_{n+1}$ line shifts left each stage in Figure 1; there comes a *recall moment*, after which $\Omega_{n-1} \geq \bar{w}_{n+1}$, when Sam strikes or quits/recalls. Summarizing:

Lemma 3 (Recall Moment) *Sam's choice margin shifts from strike or pass, to strike or quit/recall. Search then ends — sooner for a greater search cost c or quit payoff u.*

[10] The additive expression relies on no discounting. This is justified for any search that does not last months. This additive structure does not arise with a discount factor $\beta < 1$, for then (1) becomes $\bar{w}_n = \beta[-c + \bar{w}_n F_n(\bar{w}_n) + \int_{\bar{w}_n}^{\infty} w dF_n(w)]$. So if $\bar{w}_n = x_n + \zeta(c, x_n)$, then ζ solves $[x_n + \zeta](1-\beta)/\beta + c = \int_{\zeta}^{\infty}[1 - H(s)]ds$. If $\beta \in (0,1)$, then $\zeta(c, x_n)$ falls in x_n, invalidating the additive expression in Lemma 2.

Proof: Since Sam either strikes or passes in stage n when $\Omega_{n-1} < x_{n+1} + \zeta(c)$, the recall moment is at least n with probability $P(\max_{j\leq n-1}\{u, x_j + \mathcal{Z}_j\} < x_{n+1} + \zeta(c))$. This chance falls in c and u, and so the transition time falls stochastically in c and u. □

The *value function* $V_n(\Omega_n)$ at stage n is the maximum payoff when the best option so far is Ω_n. This is finite with finitely many options whose prize distributions have finite moments (by log-concavity). Clearly, $V_N(\Omega_N) = \Omega_N$. For any $n < N$, backward induction logic recursively yields value functions $V_{n-1}, \ldots, V_1$ via the Bellman equation:

$$V_n(\Omega_n) = \max\left\{\Omega_n, -c + V_{n+1}(\Omega_n)F_{n+1}(\Omega_n) + \int_{\Omega_n}^{\infty} V_{n+1}(z)dF_{n+1}(z)\right\}. \tag{3}$$

While the recursion involves all known factors, *Sam optimally stops at stage n seeing only the next known factor x_{n+1}*; this is the dynamic programming one-stage look-ahead property for optimal search.[11] For the reservation prize $\bar{w}_{n+1}$ depends only on F_{n+1} in (1).

We now pursue a novel result outside the scope of Weitzman (1979). In a stationary setting, the continuation value and reservation wage coincide. But in a nonstationary model such as we have, recall furnishes valuable insurance, and lifts the value above the diagonal: $V_n(u) > u$ on $(u, \bar{w}_{n+1})$.[12] Sam is increasingly emboldened by this insurance, and as his fallback improves, the value V_n increases on $(u, \bar{w}_{n+1})$, as in Figure 2. But fixing the fallback Ω, Sam's prospects dim with the passage of time for heterogeneous inside options: The falling reservation prizes $\bar{w}_{k+1}$ eventually dip below Ω.

Lemma 4 (Value Function Slope) *$V_n(\Omega_n)$ is convex in Ω_n in stages $n = 1, \ldots, N$, and $V_n'(\Omega_n)$ exists $(\bar{w}_{n+1}, \bar{w}_n)$, and is Sam's expected chance of eventually exercising Ω_n. As c rises, the slope $V_n'(\Omega)$ weakly rises, and thus so too does the eventual exercise chance.*

A strictly convex value, or increasing value slope, is consistent with the increasing probability of exercising the fallback. For a gentle insight into this useful probabilistic interpretation of the slope, notice that it trivially holds in the last period here: The value function $V_N(\Omega_N) = \Omega_N$ has unit slope, and Sam always accepts the fallback Ω_N: The acceptance chance is the slope $V_N'(\Omega_N) = 1$. Inductively, suppose that Sam eventually exercises the fallback option Ω_n at stage $n+1$ with chance $V_{n+1}'(\Omega_n)$. Differentiating the Bellman equation (3) for fallbacks $\Omega_n < \bar{w}_{n+1}$ yields $V_n'(\Omega_n) = F_{n+1}(\Omega_n)V_{n+1}'(\Omega_n)$. By

[11] We appealed to Weitzman (1979). But a proof by the one-stage look-ahead property works, as the search problem is *monotone*, i.e. if a searcher stops at stage n then he also stops at stage $n+1$.

[12] This is a possibility that arises with nonstationary search (Smith, 1999).

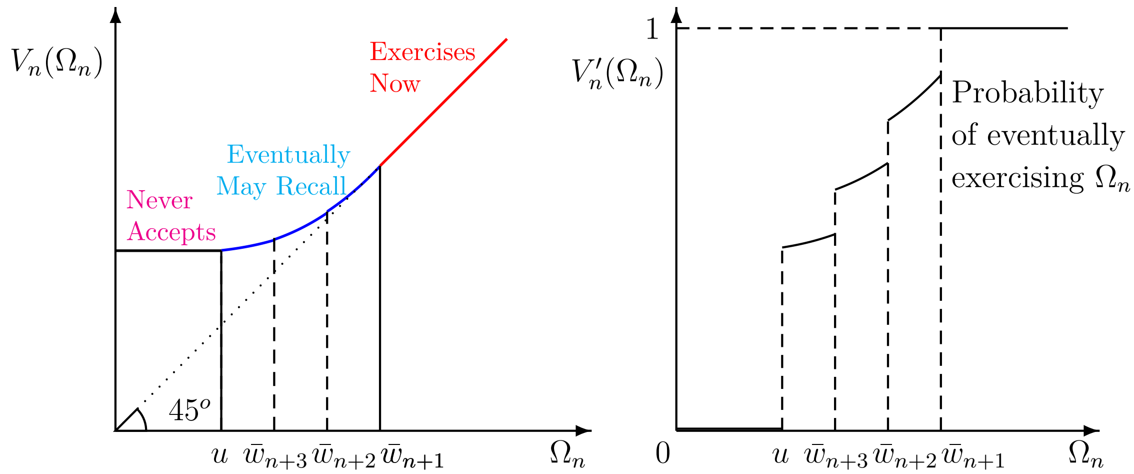

Figure 2: **Value Function and its Slope as an Eventual Exercise Chance.** We schematically plot a value V_n and its slope in the fallback $\Omega_n = \max(w_1, \ldots, w_n)$. Sam immediately exercises a prize $w \geq \bar{w}_{n+1}$, and recalls Ω_{n-1} if $n \geq 2$ iff $\Omega_{n-1} = \Omega_n \in [\bar{w}_{n+1}, \bar{w}_n)$. So V_n is constant on $(-\infty, u)$, then increasing and strictly convex on $[u, \bar{w}_{n+1})$, and finally coincides with the 45^o diagonal — and its slope is first 0, then positive and increasing, and eventually one (plotted at right). (Lemma 4)

Lemma 1, at stage n, Sam eventually exercises the fallback option Ω_n if it is the best in all stages $k \geq n$. But since by independence of hidden factors across stages,

$$P(\Omega_n \text{ best in stages } k \geq n) = P(\Omega_n \text{ best in stage } n+1) P(\Omega_n \text{ best in stages } k \geq n+1).$$

the probabilistic meaning of V'_{n+1} implies it of V'_n, as the right side is $F_{n+1}(\Omega_n) V'_{n+1}(\Omega_n)$.

4 How Does Search Change Over Time?

We now explore the changing likelihood of all search outcomes from the modeller's perspective. The known factors dictate search order (Lemma 1), but Sam's behavior reflects hidden factors too. Assume two inside options A and B, and no outside option. Say A "delays" B if Sam explores A and then B. The *delay chance* $\delta(\chi, c)$ is the probability that an option $(\mathcal{X}, \mathcal{Z})$ delays one with known factor χ — i.e., the chance that $(\mathcal{X}, \mathcal{Z})$ has a higher reservation prize, but a realized prize below $\chi + \zeta(c)$:

$$\delta(\chi, c) \equiv P\left(\{\mathcal{X} > \chi\} \cap \{\mathcal{X} + \mathcal{Z} < \chi + \zeta(c)\}\right) = \int_\chi^\infty H\left(\chi + \zeta(c) - x\right) g(x) dx. \quad (4)$$

The *participation chance* σ_1 is Sam's probability of starting search. More generally, the *n-th survival chance* σ_n is the chance that Sam's search lasts at least n stages — unconditional on all known factors. Easily, $\sigma_0 = 1 > 0 = \sigma_{N+1}$. Then σ_n is the chance that

$$\text{Sam is willing to explore some option } (x_n, \mathcal{Z}_n)\text{, i.e., } x_n + \zeta(c) > u \tag{5a}$$

$$n-1 \text{ options } (\mathcal{X}', \mathcal{Z}') \text{ delay option } n\text{, i.e., obey } \mathcal{X}' + \zeta(c) > x_n + \zeta(c) > \mathcal{X}' + \mathcal{Z}' \tag{5b}$$

$$\text{the other } N-n \text{ options have known factors below } x_n \tag{5c}$$

Events (b) and (c) have probabilities $\delta(x_n, c)^{n-1}$ and $G(x_n)^{N-n}$. Integrating the binomial probability of (a)–(c) across known factors x_n and all N options, prospective independence yields

$$\sigma_n = N\binom{N-1}{n-1}\int_{u-\zeta(c)}^{\infty} \delta(x_n, c)^{n-1} G(x_n)^{N-n} g(x_n) dx_n. \tag{6}$$

Easily, the survival chance σ_n falls in the search cost c and the outside option payoff u.[13]

We pursue results unique to the nonstationary setting — how search changes over time. Let $\mathcal{S}_n$, $\mathcal{Q}_n$, and $\mathcal{E}_n$ be the respective *stopping*, *quitting*, and *exercising chances*, conditional on entering stage n. Then $\mathcal{S}_n = \mathcal{Q}_n + \mathcal{E}_n$, since Sam either quits or exercises an inside option after stopping. Likewise, if $\mathcal{K}_n$ and $\mathcal{R}_n$ are the respective chances of *striking* the current option or *recalling* an explored option at stage-n, then $\mathcal{E}_n = \mathcal{K}_n + \mathcal{R}_n$.

From the modeler's perspective, predicting Sam's behavior is complicated because continued search signals higher known factors to the modeler. But given log-concavity, this "selection effect" cannot overwhelm the direct effect. To see this, assume just two options. The expected gap $E[\mathcal{X}_1 - \mathcal{X}_2 | \mathcal{X}_1 = x_1]$ increases in x_1 if G is log-concave. So, unconditional on $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{X}_2$, Sam enters stage 2 less often with a higher known factor x_1.

Lemma 5 *Given that Sam hits stage n, the known factor $\mathcal{X}_n$ stochastically rises in the number N of options, search cost c, and outside option u, and falls in the stage n.*

By exploiting properties of log-concave random variables, we prove in §C.2 that as the cost c increases, the known factor $\mathcal{X}_n$ stochastically increases more than $\mathcal{X}_{n+1}$ — conditional on hitting stage n. As a result, the gap between known factors $\mathcal{X}_n - \mathcal{X}_{n+1}$ stochastically increases, and Sam stops sooner. Equivalently, the survival chance σ_n falls in c, but the next survival chance σ_{n+1} falls proportionately more than σ_n falls; this inflates the stopping hazard rate $\mathcal{S}_n \equiv 1 - \sigma_{n+1}/\sigma_n$. All told, while Sam definitely stops sooner with a higher search cost c, this stochastic prediction is subtle for the modeler.

[13] The lower bound $u - \zeta(c)$ of the integral (6) rises in c by (2), and the delay chance $\delta(x, c)$ falls in c by (2)–(4). More easily, *the survival chance σ_n falls in the outside option payoff u*, as $u - \zeta(c)$ rises in u.

For conditional on arriving at a later stage, the fallback payoff is stochastically worse, and the outside option more inviting. We show that the first direct effect dominates this selection effect for $\mathcal{R}_n$ and $\mathcal{E}_n$, by log-concavity of G and H.

Theorem 1 (Search Intensifies) *Sam's conditional recall and exercise chances, $\mathcal{R}_n$ and $\mathcal{E}_n$, rise in the stage n. The quitting chance $\mathcal{Q}_n$ rises in n for small search costs $c > 0$.*

All told,[14] the chance that Sam stops search increases as time passes.[15]

Not only can we prove that the recall probability increases with each stage, but we can also speak to *which option Sam recalls.* While earlier options have larger known factors, they have been passed over more often; this offers more damning selection evidence of their hidden factors. Yet surprisingly, the earliest options are most likely sought after:

Theorem 2 (Older Options are Recalled More Often) *If Sam explores option n, then the chance that he recalls any prior option $j < n$ falls in j, for all $n = 3, \ldots, N$.*

Proof: The elegant argument is included here. If Sam explores option n, his payoff from any prior option is below the cutoff $\bar{w}_n = \chi_n + \zeta(c)$, or search would have stopped. By the Markov property of order statistics,[16] the joint distribution of the known and hidden factors for the first $n-1$ options equals that of $n-1$ i.i.d. draws $(\mathcal{X}, \mathcal{Z})$ from (G, H), *conditional* on the known ranking $\mathcal{X} > \chi_n$ and the selection effect $\mathcal{X} + \mathcal{Z} < \chi_n + \zeta(c)$. If $\mathcal{X} = \chi > \chi_n$ is the realized known factor of any prior option, its payoff $W \equiv \chi + \mathcal{Z}$ so has the cdf:

$$P(W \leq w | W < \chi_n + \zeta(c)) = \frac{H(w - \chi)}{H(\chi_n + \zeta(c) - \chi)}. \tag{7}$$

As $w < \chi_n + \zeta(c)$, this cdf of W falls in χ by log-concavity of H, and so W stochastically increases in χ — namely, the payoffs of earlier options are stochastically ranked.[17] Since this ordering holds for all $\mathcal{X}_n$ realizations, it is also holds unconditional on $\mathcal{X}_n$. □

[14]While we cannot prove it, the striking and quitting chances $\mathcal{K}_n$ and $\mathcal{Q}_n$ are each U-shaped in n in simulations. Let the known and hidden factors $\mathcal{X}$ and $\mathcal{Z}$ (resp.) be Gaussian $\mathcal{N}(0, \alpha^2)$ and $\mathcal{N}(0, 1-\alpha^2)$. If $(\alpha, c, u, N) = (0.4, 0.01, 1, 7)$, then $\mathcal{K}_n$ is falling and then rising in n, while if $(\alpha, c, u, N) = (0.4, 0.1^{-10}, 4, 5)$, then $\mathcal{Q}_n$ falls from stage 1 to stage 2 and then increases for all higher n.

[15] We can also show (proof available upon request) a few other predictions. For instance, the striking chance $\mathcal{K}_n$ increases in c. Also, as u increases, the chances $\mathcal{K}_n$ and $\mathcal{R}_n$ fall, and $\mathcal{Q}_n$ rises. Finally, the quitting and recall chances $\mathcal{Q}_n$ and $\mathcal{R}_n$ can be hump shaped in the search cost c due to selection effects.

[16]Theorem 2.4.1 (*The Markov Property*) in Arnold, Balakrishnan, and Nagaraja (1992): Let $X_{1:n} \geq X_{2:n} \geq \cdots \geq X_{n:n}$ be order statistics of a random sample $X_1, X_2, \ldots, X_n$ from a population with cdf F and pdf f. Given $X_{i:n} = x_i$, the distribution of $X_{j:n}$, for $j < i$, is the same as that of the j-th order statistic of an $(n-i)$ sample from a population with distribution F truncated at the left by x_i.

[17]One might intuitively expect that more recent options are recalled more often, as Sam a priori thought exploring them worthwhile. Our model yields such recall behavior when the hidden factor cdf H is log-convex. For then (7) increases in χ, and thus payoffs of recent options are stochastically higher.

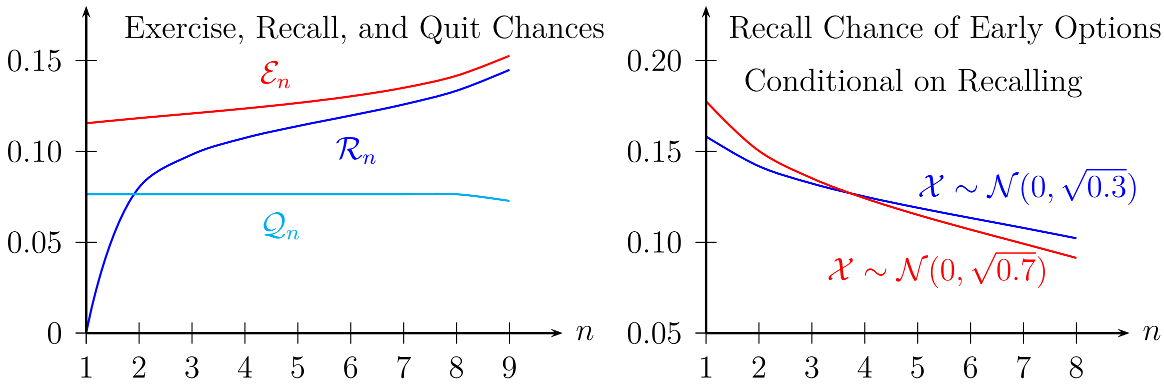

Figure 3: **Simulations.** Left: By Theorem 1, the hazard rates of recall $\mathcal{R}_n$ and exercising an inside option $\mathcal{E}_n \equiv \mathcal{R}_n + \mathcal{K}_n$ rise in n (here $\mathcal{X} \sim \Gamma(1.2, 2)$, $\mathcal{Z} \sim \Gamma(2.8, 2)$, $c = 0.2$, $N = 10$, $u = 0$). Right: By Theorem 2, Sam recalls earlier options more often, and so recall probabilities fall in n — here, from stage 9. The chance of recalling the earliest options rises in the dispersion of $\mathcal{X}$ (here $\mathcal{Z} \sim \Gamma(2.8, 2)$, $c = 0.1$, $N = 10$, $u = -10$).

5 How Long Does Search Last?

In traditional stationary search, when the prize distribution grows riskier or shifts up, search duration may rise or fall: If high prizes are more likely, Sam grows more ambitious, since the search optionality value $\zeta(c)$ in (2) rises: The stopping hazard rate may rise or fall. Does search duration increase in any general distribution shift? We claim yes.

Assume a zero fixed factor. Then (2) describes the optimal search reservation prize $\zeta(c)$. Let H_A and H_B be the respective cdfs of hidden factors $\mathcal{Z}_A$ and $\mathcal{Z}_B$. Call $\mathcal{Z}_B$ *more disperse* than $\mathcal{Z}_A$ if any two quantiles of H_B are further apart than those of H_A, i.e.

$$H_B^{-1}(\alpha'') - H_B^{-1}(\alpha') \geq H_A^{-1}(\alpha'') - H_A^{-1}(\alpha') \qquad \text{for all } 0 < \alpha' \leq \alpha'' < 1. \tag{8}$$

As the quantile function H_B^{-1} is globally steeper than H_A^{-1}, and each is differentiable, the densities must be oppositely ranked: $h_B(H_B^{-1}(\alpha)) \leq h_A(H_A^{-1}(\alpha))$ for all $\alpha \in (0, 1)$.[18,19]

Call *search duration* τ the mean number of periods searching. As the mean of a geometric distribution, it is the reciprocal of the *stopping probability* $S(c) = 1 - H(\zeta(c))$. So we change variables in the Bellman equation (2) from the prize z to its cdf $\alpha = H(z)$. If its inverse $z = H^{-1}(\alpha)$ is the *quantile function*, then $dz = dH^{-1}(\alpha) = [\partial H^{-1}(\alpha)/\partial\alpha]d\alpha$.

[18] The dispersion order differs from first- or second-order stochastic dominance since it is location free, e.g. $N(\mu, \sigma)$ has the same dispersion for all μ. But a mean-preserving increase in dispersion implies a mean-preserving spread. See Shaked and Shanthikumar (2007) (SS) for a thorough review.

[19] For this ensures a single crossing property: if the cdf's coincide, H_A increases faster than H_B.

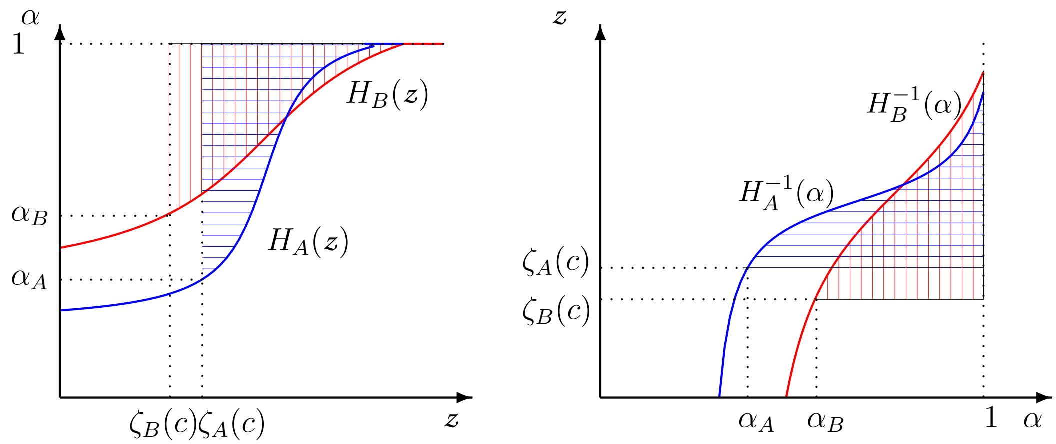

Figure 4: **Search Duration Rises in Prize Dispersion**. Optimally, the area above a cdf H right of its cutoff $\zeta(c)$ equals the search cost c. If H_B is more dispersed than H_A, its inverse quantile plot H_B^{-1} is steeper than H_A^{-1} (at right). So the (shaded) quantile slope weighted survivor integral (9) is greater, for any cutoff z. So $\zeta_A(c) > \zeta_B(c)$ and we argue that the conditional stopping chance is higher: $\alpha_B = H_B(\zeta_B(c)) > H_A(\zeta_A(c)) = \alpha_A$.

So (2) gives:

$$\int_{1-S(c)}^{1} (1-\alpha)\frac{\partial H^{-1}(\alpha)}{\partial \alpha} d\alpha = \int_{\zeta(c)}^{\infty} [1-H(z)]dz = c. \tag{9}$$

In other words, search cost equals the expected survivor probability w.r.t. the quantile distribution. By (9), the stopping hazard rate $S(c)$ falls if the quantile function H^{-1} everywhere grows steeper, so that $\partial H^{-1}(\alpha)/\partial\alpha$ everywhere rises (see Figure 4).

Theorem 3 (Stationary Search) *Search duration τ rises in prize dispersion if $\mathcal{X}\equiv 0$.*

Table 1 shows how applicable is dispersion; one parameter scales duration in each class.

We now offer an alternative economic argument that duration increases in dispersion, that is more restrictive, but allows us to parse the stopping chance derivative substitution and income effects (by analogy to consumer theory). Assume that we can smoothly index the prize pdf h_t and cdf H_t of $\mathcal{Z}$ by a dispersion index $t \in \mathbb{R}$. Differentiating the FOC $\int_{\zeta_t}^{\infty}[1-H_t(z)]dz \equiv c$ in t yields $-[1-H_t(\zeta_t)]\dot{\zeta}_t = \int_{\zeta_t}^{\infty} \dot{H}_t(z)dz$, where $\zeta_t \equiv \zeta_t(c)$ is the optionality. With this $\dot{\zeta}_t$ formula, the search stopping chance $[1-H_t(\zeta_t)]$ derivative is:

$$\frac{d[1-H_t(\zeta_t)]}{dt} = -h_t(\zeta_t)\dot{\zeta}_t - \dot{H}_t(\zeta_t) = \frac{h_t(\zeta_t)}{1-H_t(\zeta_t)}\int_{\zeta_t}^{\infty} \dot{H}_t(z)dz - \dot{H}_t(\zeta_t) \tag{10}$$

The stopping chance falls in dispersion t if the first rising standards term ("substitution effect") overwhelms the second term accounting for more high prizes ("income effect").

Easily, the stopping chance derivative (10) is negative iff $\int_{\zeta_t}^{\infty} \dot{H}_t(z)dz / \int_{\zeta_t}^{\infty} h_t(z)dz < \dot{H}_t(\zeta_t)/h_t(\zeta_t)$ on $[\zeta_t, \infty)$. In a basic ratio inequality, this holds if the fraction $\dot{H}_t(z)/h_t(z)$ strictly falls in z. If $a = H_t(z)$, consider the slope of the quantile function $z = H_t^{-1}(a)$. As $\partial H_t^{-1}(a)/\partial t = -\dot{H}_t(z)/h_t(z)$, it suffices that $\partial H_t^{-1}(a)/\partial t$ rises in a, or $\partial^2 H_t^{-1}(a)/\partial t \partial a > 0$ for all a, or $\partial H_t^{-1}(a)/\partial a$ rises in t, for all a. So *the stopping chance $1 - H_t(\zeta_t(c))$ falls in t if the quantile function $H_t^{-1}(a)$ steepens in t* — i.e. dispersion "smoothly" increases.

Mean-preserving spread (MPS) allows no search duration conclusion.[20] For if $\mathcal{Z}$ has cdf $H(z) = [(-1+1/\gamma)/z]^{\gamma}$ on $(-\infty, -1+1/\gamma]$, for $\gamma > 1$, then (see §C.3) search duration falls as small $\gamma > 1$ falls, but $\mathcal{Z}$ spreads but is not more disperse as γ falls on $(1, 2)$.

In our two factor search model, a stronger condition is needed to guarantee that Sam quits sooner. Since exploring an option with a known factor χ pays more than quitting when $\chi + \zeta(c) > u$, it suffices that $\chi + \zeta(c)$ weakly rises as $\mathcal{Z}$ grows more disperse. If $\mathcal{Z}_B$ is more disperse than $\mathcal{Z}_A$, call $\mathcal{Z}_B$ a *mean-enhancing dispersion* of $\mathcal{Z}_A$ if $E[\mathcal{Z}_B] \geq E[\mathcal{Z}_A]$. In this case, the associated search optionality values rank $\zeta_B(c) \geq \zeta_A(c)$.[21] Hence, the cutoff distribution $\mathcal{X} + \zeta(c)$ rises stochastically with a mean-enhancing dispersion of $\mathcal{Z}$.

Theorem 4 (Hidden Factors) *A mean enhancing dispersion in the hidden factor $\mathcal{Z}$ raises all survival chances σ_n, for $n = 1, 2, \ldots$, and so the participation chance σ_1 and search duration τ. The recall moment is later with a mean-preserving dispersion for $\mathcal{Z}$.*

The search duration claim follows from the well-known formula $\tau = \sum_{n=1}^{N} \sigma_n$.

In contrast to Theorem 4, greater dispersion of the known factor abbreviates search. For the order statistics $\{\mathcal{X}_n\}$ drop faster, and Sam stops sooner, as waiting is less inviting.

Theorem 5 (Known Factors) *If the known factor $\mathcal{X}$ grows more disperse and also falls stochastically, then every survival chance σ_n falls, as does the search duration.*

As $\mathcal{X}_n$ has the distribution of $G^{-1}(U_n)$, for the nth uniform $[0,1]$ order statistic U_n, the *order statistic gap* $\mathcal{X}_n - \mathcal{X}_{n+1} \sim G^{-1}(U_n) - G^{-1}(U_{n+1})$ increases stochastically in the dispersion of $\mathcal{X}$ by (8), reducing the survival chances σ_n. But since dispersion raises

[20] Keane, Todd, and Wolpin (2011) claimed that mean-preserving spreads raised search duration. For see that this claim is false, let $P(\mathcal{Z}_A = 1) = P(\mathcal{Z}_A = 2) = P(\mathcal{Z}_A = 3) = 1/3$. If $c = 1/4$, then $\zeta_A(c) = 3$ by (2). Search ends with chance $P(\mathcal{Z}_A \geq 3) = 1/3$. Spread $\mathcal{Z}_A$ to $\mathcal{Z}_B$, where $P(\mathcal{Z}_B = 1) = P(\mathcal{Z}_B = 3) = 1/2$. Now, $\zeta_B(c) = 3$ by (2), and $P(\mathcal{Z}_B \geq 3) = 1/2$. Search ends with a higher chance. But if $c = 1/2$, then $\zeta_A(c) = 2$ and $P(\mathcal{Z}_A \geq 2) = 2/3$, while $\zeta_B(c) = 3$ and $P(\mathcal{Z}_B \geq 3) = 1/2$. Search duration increases.

[21] The dispersion order asserts that H_B^{-1} is steeper than H_A^{-1}. If H_A and H_B have the same mean, H_A single crosses H_B, and so $\mathcal{Z}_B$ is a MPS of $\mathcal{Z}_A$ (Diamond and Stiglitz, 1974). Then $\zeta_B(c) \geq \zeta_A(c)$ by (2). If $\mathcal{Z}_B$ has a higher mean than $\mathcal{Z}_A$, then $\mathcal{Z}_B$ is a mean-preserving increase in dispersion of $\mathcal{Z}_A$ and a right distribution shift, as the dispersion order is location free. Each shift lifts $\zeta_B(c)$ above $\zeta_B(c)$.

Distribution	cdf	Support	More Disperse if	Thin tail?
Exponential	$1-e^{\lambda z}$	$[0,\infty)$	$\lambda\uparrow$	No
Gamma	$\frac{1}{\Gamma(k)}\gamma(k,z/\theta)$	$(0,\infty)$	$\theta\uparrow$	No
Gaussian	$\frac{1}{2}+\frac{1}{2}\text{erf}\left(\frac{z-\mu}{\sigma\sqrt{2}}\right)$	$(-\infty,\infty)$	$\sigma\uparrow$	Yes
Gumbel	$e^{-e^{-(z-\mu)/\beta}}$	$(-\infty,\infty)$	$\beta\uparrow$	No
Logistic	$1/\left(1+e^{-\frac{z-\mu}{s}}\right)$	$(-\infty,\infty)$	$s\uparrow$	No
Uniform	$(z-a)/(b-a)$	$[a,b]$	$a\downarrow$ or $b\uparrow$	Yes

Table 1: **Dispersion for Logconcave Probability Distributions.** The last column flags if recall chances vanish for this hidden factor distribution as $N\to\infty$ (Theorem 7).

the probability mass at the left- and right-tail of the random variable $\mathcal{X}$, it might raise Sam's participation chance, i.e., the chance that $\mathcal{X}_1+\zeta(c)\geq u$, and thereby increase σ_n. This countervailing participation effect vanishes if $\mathcal{X}$ stochastically falls.

Is search in our two factor model faster than purely random search? Specifically, is search duration lower if we add a random known factor $\mathcal{X}$? The answer depends on the outside option u. With no known factor, the reservation prize is constant at $\zeta(c)$, by Lemma 2. For all larger outside options $u>\zeta(c)$, as Sam never participates, a random known factor $\mathcal{X}$ can only increase search duration. But search here is otherwise faster:

Corollary 1 *Search ends sooner with a known factor iff search occurs: $\zeta(c)>u$.*

6 How Does Search Change with More Options?

A classic measure of an improved search environment is the number of options. Ease of web applications and web search has ensured that N has risen. In a stationary search model, one never quits or recalls — chances are constant at $\mathcal{Q}=\mathcal{R}=0$ — and striking occurs at a constant hazard rate $\mathcal{K}=1-H(\zeta(c))$. But with a finite number N of options, optimal search behavior evolves because the known factors $\mathcal{X}_1^N\geq\mathcal{X}_2^N\geq\cdots$ stochastically rise in N, by Lemma 5. For whether one stops this period or next depends on the order statistic gaps $\mathcal{X}_n^N-\mathcal{X}_{n+1}^N$. As N increases, these gaps stochastically weakly shrink, and this spurs search, by depressing quitting, striking, and recall hazard rates.

Theorem 6 (More Options) *The quitting, striking and recalling chances $\mathcal{Q}_n^N$, $\mathcal{K}_n^N$, and $\mathcal{R}_n^N$ in any stage n all weakly fall in the number of options N. Search duration rises with additional options, and the recall moment happens later.*

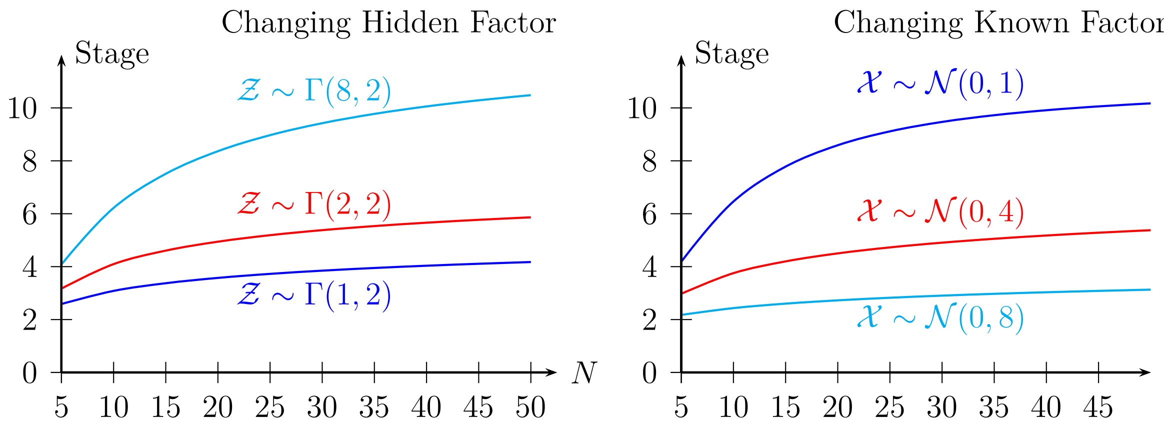

Figure 5: **Search Duration Simulation.** Assume $c = 0.2$ and $u = -\infty$. Left: By Theorem 4, duration rises in the hidden factor dispersion (with $\mathcal{X} \sim \mathcal{N}(0, 2.8)$). Right: By Theorem 5, duration falls in the dispersion of known factor $\mathcal{X}$ (with $\mathcal{Z} \sim \Gamma(1.2, 2)$).

In the knife-edge case, we have (¶): if $\mathcal{X}$ has an exponential distribution with mean λ, then the order statistic gap $\mathcal{X}_n^N - \mathcal{X}_{n+1}^N$ is also exponentially distributed, but with a mean $n\lambda$, invariant to N (Pyke, 1965). The hazard rates are thus constant in N.

Obviously, Sam's welfare rises in N, as extra options can be ignored.[22] But Theorem 6 asserts a subtler fact that search duration optimally increases in N. Nowadays, e.g., many firms receive far more web applications for every position than in pre-web days of yore, and thus positions may well remain unfilled longer.[23] Or, those searching for mates online can expected to remain unmatched much longer, with the plethora of options.

Next, consider the limiting behavior as the number of options N explodes. Does the standard stationary search model reasonably predict Sam's behavior in the limit? It should come as no surprise that Sam never quits in this limit — since the fallback option is eventually dominated by *some* inside option. But the recall probability need not vanish. As noted above in (¶), if $\mathcal{X}$ is exponentially distributed, order statistic gaps are constant, and thus Sam is strictly more likely to strike or recall in every period than in the stationary search model. The striking and recall hazard rates converge to their stationary limits only if all top order statistic gaps $\mathcal{X}_n^N - \mathcal{X}_{n+1}^N$ vanish as $N \uparrow \infty$. This happens[24]

[22] A formal analysis of Toffler's overchoice would obviously have to relax this assumption.

[23] Using CPS data, Kuhn and Skuterud (2004) find that web job search does not increase the job-finding rate. Martellini and Menzio (2018) likewise explain the stability of the unemployment rate despite major information technology improvements. They show that a more efficient matching function might not reduce the unemployment rate in the Diamond-Mortensen-Pissarides model. We model the nonstationary search process explicitly and so provide a clear foundation for their matching process. An increase in N naturally captures firms receiving more applications as information technology improves.

[24] As the hazard rate $g/[1-G]$ is non-decreasing, when $\lim_{x \uparrow G^{-1}(1)} g(x)/[1-G(x)]$ exists and is positive. A thin tail means $\lim_{x \uparrow G^{-1}(1)} g(x)/[1-G(x)] < \infty$. Table 1 lists distributions with thin tails.

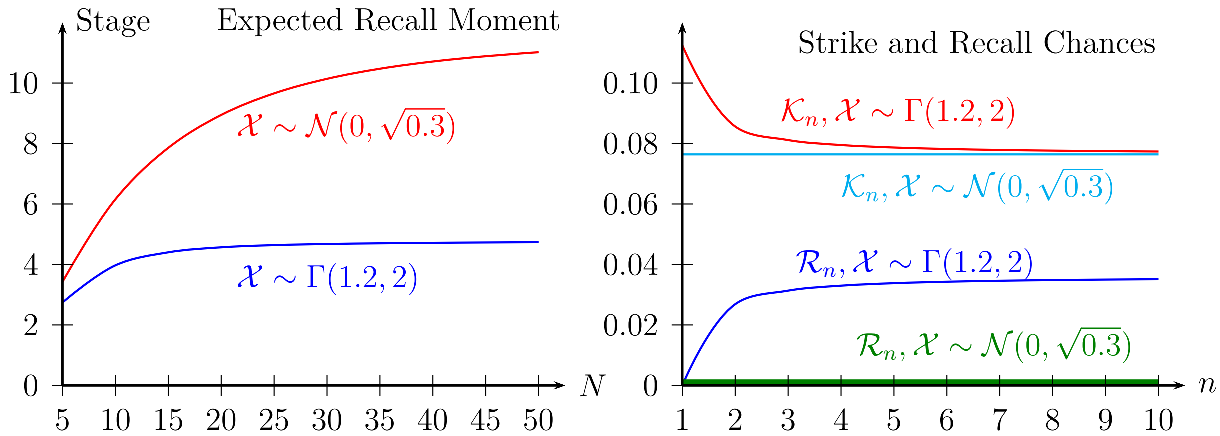

Figure 6: **Number of Options Simulation.** Left: The mean recall moment increases in N, by Theorem 6. Both distributions $\mathcal{X} \sim \mathcal{N}(0, \sqrt{0.3})$ and $\mathcal{X} \sim \Gamma(1.2, 2)$ have a common $Var(\mathcal{X}) = 0.3$. Right: By Theorem 7, as $N \to \infty$, the recall chance $\mathcal{R}_n$ vanishes when $\mathcal{X}$ has a fat tail (here, for $\mathcal{X}$ Gaussian), but otherwise is strictly positive (here, the Gamma distribution). Both panels assume $\mathcal{Z} \sim \Gamma(2.8, 2)$, $c = 0.2$ and $u = -\infty$.

if the distribution G has a *thin (right) tail*, namely, if $\lim_{\chi \uparrow G^{-1}(1)} g(\chi)/[1-G(\chi)] = \infty$. This excludes our knife-edge exponential case, given the constant hazard rate $n\lambda > 0$.

Theorem 7 (Limit) *Fix a stage n, and let $N \to \infty$. Then $\mathcal{Q}_n^N \to 0$. If G has a thin tail, then $\mathcal{R}_n^N \to 0$ and $\mathcal{K}_n^N \to 1 - H(\zeta(c))$. Otherwise, $\mathcal{R}_n^N \to \mathcal{R}_n^\infty > 0$ and $\mathcal{K}_n^N \to \mathcal{K}_n^\infty > 1 - H(\zeta(c))$, where $\mathcal{R}_n^\infty + \mathcal{K}_n^\infty < 1$. The limit recall probability $\mathcal{R}_n^\infty$ rises in $\mathcal{X}$'s dispersion.*

When the known factor lacks a thin tail, optimal behavior with a vast number of options differs much from the infinite horizon search model. Sam recalls with a boundedly positive chance. Sam also strikes more often than justified by the hidden noise. The reason is that the gaps between consecutive known factors do not vanish; this provides an extra incentive to strike now — as next period is worse than this one.

7 Conclusion

We develop and characterize a tractable twist on the benchmark stationary search model to capture practical economic settings, like web search, with *a priori* distinct options. We assume finitely many options whose payoffs are the sum of known and hidden factors. This generates random families of Weitzman (1979) search models with selection effects.

1. With logconcave distributions, search intensifies over time: quitting, recall and exercise chances all increase. Also, if recall occurs, older options are recalled more often. This is the first robust search model with clear predictions for positive recall rates.

2. We have resolved a basic but long outstanding open problem in search theory — what distribution shifts increase search duration? For the benchmark stationary search model, increasing prize dispersion prolongs search. In our richer setting, dispersion of the hidden factor prolongs search, and dispersion of the known factor truncates search.

3. Finally, we show that stationary search poorly predicts behavior in the finite real world: Recall only vanishes in the many option limit if the known factor has a thin tail.

Our two factor model speaks to many economic applications. For example, online matching has improved over time (Regnerus, 2017). If new technology raises signal precision, then known factors grow more disperse, and matching speeds up (Theorem 5). But if it instead more cheaply yields the same quality signal, then people grow pickier.

A Duration in the Stationary Job Search Model

Search duration rises in prize dispersion in our two factor model. But the result is more nuanced in McCall's classic 1970 infinite horizon job search model. Given discount factor $\beta < 1$, wage density $f = F'$, and search cost c, the reservation wage $\bar{w}(c)$ obeys:

$$(1-\beta)\bar{w}(c) = -c + \beta \int_{\bar{w}(c)}^{\infty} [1 - F(w)]dw. \tag{11}$$

Differentiate the Bellman equation (11) to get $\bar{w}'(c) = -[1 - \beta + \beta(1 - F(\bar{w}(c)))]^{-1}$. Then

$$\frac{\partial[1 - F(\bar{w}(c))]}{\partial c} = -f(\bar{w}(c))\bar{w}'(c) = \frac{f(\bar{w}(c))}{1 - \beta + \beta(1 - F(\bar{w}(c)))}. \tag{12}$$

If F_2 is more dispersed than F_1, then $F_2(\bar{w}_2(c)) = F_1(\bar{w}_1(c))$ implies $f_2(\bar{w}_2(c)) \leq f_1(\bar{w}_1(c))$. Ignoring the middle term of (12), this single crossing property says $\partial[1 - F_1(\bar{w}_1(c))]/\partial c \geq \partial[1 - F_2(\bar{w}_2(c))]/\partial c$ if $F_2(\bar{w}_2(c)) = F_1(\bar{w}_1(c))$, and so $[1 - F_2(\bar{w}_2(c))] \gtrless [1 - F_1(\bar{w}_1(c))]$ as $c \lessgtr \bar{c}$, for some $\bar{c}$. If $\bar{c} < 0$, then $c < 0$ is negative unemployment benefits. Assume wage dispersion rises. Then the stopping chance $[1 - F(\bar{w}(c))]$ rises for low costs c, and falls for higher c. So while search duration rises for undiscounted search (Theorem 3), *duration in stationary discounted wage search rises for high enough search costs.*

For some popular parametric distributions, dispersion reflects scaling wages to aW, where $a > 1$ (eg. Gaussian or Gamma). Notably, in a stationary search model with no outside option, this induces the same behavior as scaling search costs to c/a. With unemployment benefits, so $c < 0$, a fall in unemployment benefit to c/a naturally reduces search duration. So duration rises in the scale parameter iff $c > \bar{c} = 0$.

B Additional Topic: How Does Search End?

We turn to the *quitting chance* q — the probability that Sam either does not participate, or does, but eventually chooses the outside option. In a stationary search environment, if one is willing to search, then one never quits: $q = 0$. But we capture how one may fail to get hired in a job search, or to cut a deal in consumer search. For instance, after using an online shopping search engine, a searcher eventually buys with chance $1 - q$.

Sam explores a ranked option $(\chi, \mathcal{Z})$ only if he does not quit before, and so iff $\bar{w} > u$, by Lemma 2, and so iff $\chi > u - \zeta(c)$. With one inside option, Sam quits either because:

(*a*) the option is dominated: nonparticipation chance $q_0 = G(u - \zeta(c))$ of $\mathcal{X} + \zeta(c) < u$

(*b*) Sam explores it, then quits: delay chance $q_1 = \delta(u - \zeta(c), c)$ of $u - \zeta(c) < \mathcal{X} \leq u - \mathcal{Z}$

Define $\pi(\chi, c) \equiv G(\chi) + \delta(\chi, c)$. The *quitting chance* with one inside option is $q = q_0 + q_1 \equiv \pi(u - \zeta(c), c)$. Next posit N inside options. By prospective independence, the probability that Sam explores exactly $n \leq N$ inside options and then quits equals:

$$q_n = \binom{N}{n} \delta(u - \zeta(c), c)^n G(u - \zeta(c))^{N-n}. \tag{13}$$

Since $\pi(u - \zeta(c), c) \equiv \delta(u - \zeta(c), c) + G(u - \zeta(c))$, the binomial theorem and (13) yield a quitting chance:

$$q = \sum_{n=0}^{N} q_n = \pi(u - \zeta(c), c)^N. \tag{14}$$

Since $\pi(u - \zeta(c), c)$ rises in u and c, given (4), intuitively: *the quitting chance q rises in the outside option u and search cost c, and falls in the number of options N*. But subtly, more dispersion of the hidden factor accelerates quitting iff the outside options is low.

Theorem 8 (Quitting Chance) *If the hidden factor $\mathcal{Z}$ dispersion rises, then q rises iff $u < \bar{u}$, for some $\bar{u} \in \mathbb{R} \cup \pm\infty$. If it is a mean preserving dispersion, then $|\bar{u}| < \infty$.*

With just one inside option $(\mathcal{X}, \mathcal{Z})$, Sam quits if $\mathcal{X} + \min(\mathcal{Z}, \zeta(c)) \leq u$; so he doesn't participate $(\mathcal{X} + \zeta(c) \leq u)$ or declines the inside option $(\mathcal{X} + \mathcal{Z} < u)$. For a mean preserving dispersion of $\mathcal{Z}$, the lower $\mathcal{Z}$ quantiles fall and $\zeta(c)$ rises. So $P(\mathcal{X} + \min(\mathcal{Z}, \zeta(c)) \leq u)$ rises for small u and falls for large u.

After a mean preserving dispersion, the quitting chance rises for low fallbacks, and otherwise falls (Theorem 8). Consider the classic search models: For product search — the first case, since one buys regardless of price — dispersion leads one to quit. For job search — the second case, as one might not take a job — dispersion deters quitting.

Proof of Theorem 8: Let $\mathcal{Z}_B$ be a mean preserving dispersion of $\mathcal{Z}_A$, with respective cdfs H_B and H_A. The quitting chance is $q_a = \pi_a(u - \zeta(c), c)^N$, for $a = A, B$ by (14). It suffices that $q_B = \pi_B(u - \zeta_B(c), c) \gtreqless \pi_A(u - \zeta_A(c), c) = q_A$ as $u \lesseqgtr \bar{u}$ for some $\bar{u}$.

Let $\underline{H}_a$ be the cdf of $\min\{\mathcal{Z}_a, \zeta_a(c)\}$, for $a = A, B$. Posit $\zeta_B(c) \geq \zeta_A(c)$. If $z < \zeta_A(c)$, then $\underline{H}_B(z) - \underline{H}_A(z) = H_B(z) - H_A(z)$. This is *downcrossing* (crosses at most once from $+$ to $-$), since H_B^{-1} is steeper than H_A^{-1}. Likewise, $\underline{H}_B(z) - \underline{H}_A(z) = H_B(z) - 1 \leq 0$ for $z \in [\zeta_A(c), \zeta_B(c))$, and $\underline{H}_B(z) - \underline{H}_A(z) = 0$ for $z > \zeta_B(c)$. So $\underline{H}_B - \underline{H}_A$ is downcrossing in this case. Lastly, we similarly deduce that $\underline{H}_B - \underline{H}_A$ is downcrossing when $\zeta_B(c) \leq \zeta_A(c)$.

We can rewrite $\pi(u - \zeta(c), c) \equiv P(\min(\mathcal{Z}, \zeta(c)) \leq u - \mathcal{X})$ as:

$$\pi(u - \zeta(c), c) = \int_{-\infty}^{\infty} P(\{\min(\mathcal{Z}, \zeta(c)) \leq s\} \cap \{s = u - \mathcal{X}\}) ds = \int_{-\infty}^{\infty} \underline{H}_a(s) g(u - s) ds.$$

Since $\underline{H}_B(s) - \underline{H}_A(s)$ is downcrossing, so is $\pi_B(u - \zeta_B(c), c) - \pi_A(u - \zeta_A(c), c) = \int_{-\infty}^{\infty} [\underline{H}_B(s) - \underline{H}_A(s)] g(u - s) ds$, as g is a log-concave density (Karlin and Rubin, 1955).

Assume a mean preserving dispersion of $\mathcal{Z}$. Integrating (2) by parts, $\zeta_a(c) = -c + E[\mathcal{Z}] + \int_{-\infty}^{\zeta_a(c)} H_a(z) dz$. As this is a MPS, $\int_{-\infty}^{a} H(z) dz$ rises, and so $\zeta_B(c) > \zeta_A(c)$.[25] We rule out $\pi_A(u - \zeta_A(c), c) > \pi_B(u - \zeta_B(c), c)$ for all u. As $\pi_a(u - \zeta_a(c), c)$ is the cdf of $\mathcal{X} + \min\{\mathcal{Z}_a, \zeta_a(c)\}$, we rule out $\mathcal{X} + \min\{\mathcal{Z}_B, \zeta_B(c)\} \succ \mathcal{X} + \min\{\mathcal{Z}_A, \zeta_A(c)\}$ stochastically. This contradicts $E[\mathcal{X} + \min\{\mathcal{Z}_B, \zeta_B(c)\}] = E[\mathcal{X} + \min\{\mathcal{Z}_A, \zeta_A(c)\}]$, as $E[\mathcal{Z}_B] = E[\mathcal{Z}_A]$ and:

$$E[\min\{\mathcal{Z}_a, \zeta_a(c)\}] - E[\mathcal{Z}_a] = \int_{\zeta_n(c)}^{\infty} (\zeta_a(c) - z) dH_a(z) = \int_{\zeta_a(c)}^{\infty} [1 - H_a(z)] dz = c$$

by (2). Altogether, $\pi_B(u - \zeta_B(c), c) - \pi_A(u - \zeta_A(c), c)$ is downcrossing in u. □

C Omitted Proofs

C.1 Optimal Stopping: Proof of Lemma 4

All claims hold at stage N, when we reduce to a one-shot search problem with a fallback option: $V_N(\Omega) = \Omega$ for $\Omega < \bar{w}_N$ as the best option Ω_N is exercised. Then $V_N'(\Omega) = 1$.

Assume all claims at stage $n + 1$. Search stops at stage n if $\Omega_n \geq \bar{w}_{n+1}$. By (3), $V_n(\Omega) = \Omega$ on $[\bar{w}_{n+1}, \infty)$ and so $V_n'(\Omega) = 1$, i.e., the stopping chance. Sam searches at stage $n + 1$ if $\Omega_n < \bar{w}_{n+1}$. Then $V_n'(\Omega_n) = F_{n+1}(\Omega_n) V_{n+1}'(\Omega_n)$ by (3). Since V_{n+1}'

[25] When $\mathcal{Z}$ has full support, integration by parts requires $\lim_{z \to -\infty} zH(z) < \infty$. By l'Hopital's rule, $\lim_{z \to -\infty} zH(z) = \lim_{z \to -\infty} -z^2 h(z)$. These limits vanish, for otherwise the second moment $\int_{-\infty}^{\infty} z^2 h(z) dz$ is infinite — impossible, as log-concave densities have finite moments (An, 1997).

jumps up at $\bar{w}_N < \cdots < \bar{w}_{n+2}$, so does V_n'. Now, $1 = V_n'(\bar{w}_{n+1}+) > V_n'(\bar{w}_{n+1}-) = F_{n+1}(\bar{w}_{n+1})V_{n+1}'(\bar{w}_{n+1}-)$ as $V_{n+1}'(\bar{w}_{n+1}-) < 1$ by assumption, and $F_{n+1}(\bar{w}_{n+1}) < 1$. Then V_n' exists except at jumps $\bar{w}_N < \cdots < \bar{w}_{n+1}$. If $\Omega_n < \bar{w}_{n+1}$, then Sam enters stage $n+1$ and will recall Ω_n with chance $V_n'(\Omega_n) = F_{n+1}(\Omega_n)V_{n+1}'(\Omega_n)$. As F_{n+1} has full support and V_{n+1} is convex, $F_{n+1}V_{n+1}'$ rises, and $F_{n+1}(\Omega_n) < 1$. So V_n is strictly convex and $V_n'(\Omega) < 1$ for all $\Omega < \bar{w}_{n+1}$. The last claim is true because Sam enters stage k with best-so-far w iff $w = \Omega_{k-1} < \bar{w}_k$, and he recalls w iff $w = \Omega_k \geq \bar{w}_{k+1}$, by Lemma 1. □

Finally, $V_n(\Omega)$ *grows weakly steeper for all* $n = 1, \ldots, N$. The claim holds if $n = N$, as $V_N'(\Omega) = 1$. For $n < N$, $V_n'(\Omega) = 1$ for $\Omega \geq \bar{w}_{n+1}$ and $V_n'(\Omega) = F_{n+1}(\Omega)V_{n+1}'(\Omega) < 1$ otherwise. As $\bar{w}_{n+1}$ falls in c by (3), and $V_{n+1}'(\Omega)$ weakly rises in c, so does $V_n'(\Omega)$. □

C.2 Search Behavior Over Time

(a) **Preamble.** We prove Lemma 5 and Theorem 1 by defining Sam's interim stage n probability measure. Define the *ex ante* probability density of hitting stage n with known factor $\mathcal{X} = \chi$:

$$\eta(\chi, c, n, N) \equiv \delta(\chi, c)^{n-1} G(\chi)^{N-n} g(\chi). \tag{15}$$

By (6) and (15), the *stage-n conditional expectation* operator $E_{\mathcal{X}_n}$ is given by the cdf

$$P(\mathcal{X}_n \leq a | \text{enters stage } n) = \frac{N\binom{N-1}{n-1}\int_{u-\zeta(c)}^{a} \eta(\chi, c, n, N) d\chi}{\sigma_n} = \frac{\int_{u-\zeta(c)}^{a} \eta(\chi, c, n, N) d\chi}{\int_{u-\zeta(c)}^{\infty} \eta(\chi, c, n, N) d\chi}. \tag{16}$$

(b) Stochastic Shifts of the Known Factor in the Proof of Lemma 5

We argue that the cdf (16) *falls in* N, c *and* u, *and increases in* n. By (15), we have:

$$\frac{\partial}{\partial a} \log \left[\int_{u-\zeta(c)}^{a} \eta(\chi, c, n, N) d\chi \right] = \frac{\delta(a, c)^{n-1} G(a)^{N-n} g(a)}{\int_{u-\zeta(c)}^{a} \delta(\chi, c)^{n-1} G(\chi)^{N-n} g(\chi) d\chi}. \tag{17}$$

We repeatedly use logsupermodularity. Since $G(a)/G(\chi) > 1$ if $\chi < a$, the RHS of (17) rises in N. So the bracketed integral in (17) is LSPM in (N, a), and ratio (16) falls in N, as $a < \infty$. Since $\delta(\chi, c)$ is LSPM by Claim 1 below, $\delta(a, c)/\delta(\chi, c)$ rises in c, if $\chi < a$. Since $u - \zeta(c)$ rises in c, the RHS of (17) rises in c and in u. So $\int_{u-\zeta(c)}^{a} \eta(\chi, c, n, N) d\chi$ is LSPM in (c, a) and (u, a). Then the ratio (16) falls in c and u. Finally, $\delta(\chi, c)/G(\chi)$ falls in χ, by Claim 1, and so $\delta(\chi, c)/G(\chi) > \delta(a, c)/G(a)$ for $\chi < a$. So (17) falls in n, and the bracketed integral in (17) is LSPM in (n, a). Thus, the ratio (16) rises in n. □

Claim 1 (**Delay Chance**) *$\delta(\chi, c)$ falls in c and is LSPM; also, $\delta(\chi, c)/G(\chi)$ falls in χ.*

Proof: Put $s = a - \chi$ in (4). Then $\delta(\chi, c) = \int_0^\infty H(\zeta(c) - s)\, g(s + \chi) ds$. Then $\zeta'(c) < 0$ implies $\delta_c(\chi, c) < 0$. Since $H(\zeta(c) - s)$ and $g(s + \chi)$ are LSPM in $(\zeta(c), s)$ and $(s, -\chi)$, resp., and partial integration preserves LSPM (Karlin and Rinott, 1980), $\delta(\chi, c)$ is LSPM in $(\zeta(c), -\chi)$, and so in (c, χ). Integrating (4) by parts, the delay chance is

$$\delta(\chi, c) = -H(\zeta(c))\, G(\chi) + \int_0^\infty h(\zeta(c) - s)\, G(s + \chi) ds. \tag{18}$$

Since $G(\chi)$ is log-concave, $G(s+\chi)/G(\chi)$ falls in χ, and thus so does $\delta(\chi, c)/G(\chi)$. □

(*c*) Conditional Stopping Chance and Search Costs in the Proof of Lemma 5

We prove a claim after Lemma 5 that the *stopping hazard rate* $\mathcal{S}_n$ rises in c. By (6):

$$1 - \mathcal{S}_n \equiv \frac{\sigma_{n+1}}{\sigma_n} = \frac{(N-n)\int_{u-\zeta(c)}^\infty [\delta(\chi, c)/G(\chi)]\eta(\chi, c, n, N) d\chi}{n \int_{u-\zeta(c)}^\infty \eta(\chi, c, n, N) d\chi} = \frac{N-n}{n} E_{\mathcal{X}_n}\left[\frac{\delta(\mathcal{X}_n, c)}{G(\mathcal{X}_n)}\right] \tag{19}$$

as $\eta(\chi, c, n, N) = \delta(\chi, c)^{n-1} G(\chi)^{N-n} g(\chi)$. Then $\mathcal{S}_n$ rises in c, as $\delta(\chi, c)/G(\chi)$ falls in χ (Claim 1), $\mathcal{X}_n$ stochastically rises in c (Lemma 5), and $\delta(\chi, c)$ falls in c ((4) and $\zeta' < 0$).□

(*d*) Conditional Stopping Chances Increase: Proof of Theorem 1

The quitting hazard rate $\mathcal{Q}_n \equiv q_n/\sigma_n$ rises in the stage n if σ_n falls and q_n rises in n. Now, the survival chance σ_n that search lasts at least n stages must fall in n. Next:

Claim 2 (**Quitting Chance**) *The quitting chance q_n rises in the stage n for all small costs $c > 0$, is hump-shaped in n for intermediate c, and falls in n for all large c.*

Proof: By (13), the ratio $q_{n+1}/q_n = [(N-n)/(n+1)]\delta(u - \zeta(c), c)/G(u - \zeta(c))$ falls in $n = 0, 1 \ldots, N-1$. For if $\delta/G < 1/N$, then $q_{n+1}/q_n < 1$ for all $n = 0, \ldots, N-1$, and so q_n falls in n. Next, if $\delta/G > N$, then $q_{n+1}/q_n > 1$ for all $n = 0, \ldots, N-1$, and so q_n rises in n. Finally, if $1/N < \delta/G < N$, then q_n rises and then falls as n rises from 0 to N.

Next, we show that $\delta(u - \zeta(c), c)/G(u - \zeta(c))$ falls from ∞ to 0 as for $c \in [0, \infty)$. For (4) implies:

$$\frac{\delta(u - \zeta(c), c)}{G(u - \zeta(c))} = \int_{-\zeta(c)}^\infty H(-s) \frac{g(s+u)}{G(u - \zeta(c))} ds. \tag{20}$$

Since $\zeta'(c) < 0$, (20) falls in c, vanishing as $c \to \infty$ (for then $\zeta(c) \to -\infty$ by (2)), exploding as $c \to 0$ (for then $\zeta(c) \to \infty$, and thus $H(\zeta(c)-r) \to 1$). Hence, (20) implies:

$$\lim_{c\to 0} \frac{\delta(u-\zeta(c),c)}{G(u-\zeta(c))} = \lim_{c\to 0} \frac{\int_0^\infty H(\zeta(c)-r)g(r+u-\zeta(c))dr}{G(u-\zeta(c))} = \lim_{\zeta(c)\to\infty} \frac{[1-G(u-\zeta(c))]}{G(u-\zeta(c))} = \infty.$$

So there exists $\bar{c} > \underline{c}$ s.t. (1) $\delta(u-\zeta(c),c)/G(u-\zeta(c)) > N$ if $c < \underline{c}$, and so q_n rises in n; (2) $\delta(u-\zeta(c),c)/G(u-\zeta(c)) \in [1/N, N]$ if $c \in [\underline{c}, \bar{c}]$, and so q_n is hump-shaped in n; and (3) $\delta(u-\zeta(c),c)/G(u-\zeta(c)) < 1/N$ if $c > \bar{c}$, and so q_n falls in n. □

Consider the temporary perspective of a partially omniscient observer, who knows only one realized option (χ, z). Let $w^*(\chi, z) \equiv \min(\chi + z, \bar{w})$, for the reservation prize $\bar{w} = \chi + \zeta(c)$. Modifying the n-survival event (5a)–(5c), Sam exercises (χ, z) at stage n iff

$$w^*(\chi, z) > u \tag{21a}$$

$$\mathcal{X}' + \zeta(c) > w^*(\chi, z) > \mathcal{X}' + \mathcal{Z}' \text{ for } n-1 \text{ options } (\mathcal{X}', \mathcal{Z}') \tag{21b}$$

$$w^*(\chi, z) > \mathcal{X}'' + \zeta(c) \text{ for } N-n \text{ options } (\mathcal{X}'', \mathcal{Z}''). \tag{21c}$$

To see that this implies n-survival, posit (21). Then (5) holds if $w^*(\chi, z) = \bar{w}$. And if $w^*(\chi, z) = \chi + z < \bar{w}$, all $\mathcal{X}' + \zeta(c)$ and $\bar{w}$ exceed $\chi + z$, all $\mathcal{X}' + \mathcal{Z}'$, all $\mathcal{X}'' + \zeta(c)$, and u; so Sam explores (χ, z) and all $(\mathcal{X}', \mathcal{Z}')$. We now claim Sam exercises (χ, z) in event (21).

Claim 3 (Exercise) *Sam strikes (χ, z) if (21) and $\mathcal{X}' > \chi$ for all $n-1$ options $(\mathcal{X}', \mathcal{Z}')$. Sam recalls (χ, z) if (21) and $\mathcal{X}' \leq \chi$ for one or more of the $n-1$ options $(\mathcal{X}', \mathcal{Z}')$.*[26]

Proof: As $w^* \equiv w^*(\chi, z) > u$, Sam never quits in a one option world (by §B), and so never quits with N options. The question is which inside option does Sam exercise, and how?

Assume (21). First, (χ, z) *blocks* (*ex ante* dominates) the $N-n$ options with $w^* > \mathcal{X}'' + \zeta(c)$ — i.e. Sam never later explores them. We next show that Sam explores all $n-1$ options $(\mathcal{X}', \mathcal{Z}')$ in some order, and then exercises (χ, z). First, Sam explores (χ, z) at some stage, as no option $(\mathcal{X}', \mathcal{Z}')$ blocks it: For $\bar{w} > \mathcal{X}' + \mathcal{Z}'$, by (21).

Next, since $\mathcal{X}' + \zeta(c) > w^*$, either $\mathcal{X}' + \zeta(c) > \bar{w}$ and so Sam explores $(\mathcal{X}', \mathcal{Z}')$ before (χ, z), or $\bar{w} \geq \mathcal{X}' + \zeta(c) > \chi + z$, and so (χ, z) delays Sam. In either eventuality, (χ, z) does not block $(\mathcal{X}', \mathcal{Z}')$. Finally, as any two of the $n-1$ options $(\mathcal{X}'_A, \mathcal{Z}'_A)$ and $(\mathcal{X}'_B, \mathcal{Z}'_B)$ obey $\mathcal{X}'_A + \zeta(c) > w^* > \mathcal{X}'_B + \mathcal{Z}'_B$ by (21), no option $(\mathcal{X}'_B, \mathcal{Z}'_B)$ blocks another $(\mathcal{X}'_A, \mathcal{Z}'_A)$. So Sam eventually explores all $(\mathcal{X}', \mathcal{Z}')$, exercising (χ, z) at stage n, as $\chi + z > \mathcal{X}' + \mathcal{Z}'$.

[26]If two options have the same known factor (a measure zero event), Sam's behavior can be arbitrary.

When $\mathcal{X}' > \chi$ for $n-1$ known factors, the option (χ, z) must be the n^{th} option, and hence Sam strikes (χ, z). The last claim follows at once from (21) and the first claim.□

We first introduce the interim random variable $\mathcal{W}^* \equiv \mathcal{X} + \min(\mathcal{Z}, \zeta(c)) \equiv w^*(\mathcal{X}, \mathcal{Z})$. The interim n-exercise event (21) equals the n-survival event (5) if $w^* = \chi + \zeta(c)$, and has chance

$$\Lambda_n(w) \equiv N\binom{N-1}{n-1}\delta(w-\zeta(c), c)^{n-1}G(w-\zeta(c))^{N-n} \tag{22}$$

for $w > u$, and $\Lambda_n(w) = 0$ otherwise. Next, convoluting densities for $z = \zeta(c) - s$ and $\chi = w^* - \min(z, \zeta(c)) = w^* - \zeta(c) + \max(s, 0)$, we see that $\mathcal{W}^*$ has *ex ante* probability density

$$\phi(w^*) \equiv \textstyle\int_{-\infty}^{\infty} h(\zeta(c)-s)g(w^*-\zeta(c)+\max(s,0))ds. \tag{23}$$

Hence, the n-survival chance (6) as $\sigma_n = E_g[\Lambda_n(\mathcal{X} + \zeta(c))]$, and the n-exercise chance as

$$e_n = E_\phi[\Lambda_n(\mathcal{W}^*)] = N\binom{N-1}{n-1}\int_u^\infty \delta(w^*-\zeta(c), c)^{n-1}G(w^*-\zeta(c))^{N-n}\phi(w^*)dw^*. \tag{24}$$

Steps 1–3 use the operator $E_{\mathcal{X}_n}$ to characterize the conditional chances $\mathcal{E}_n, \mathcal{K}_n, \mathcal{R}_n$.

Step 1 (Exercise Chance Formula) *The conditional exercise chance $\mathcal{E}_n$ rises in n, and*

$$\mathcal{E}_n = 1 - H(\zeta(c)) + E_{\mathcal{X}_n}\left[\int_0^\infty h(\zeta(c)-s)\frac{g(s+\mathcal{X}_n)}{g(\mathcal{X}_n)}ds\right]. \tag{25}$$

The lead $1 - H(\zeta(c))$ term is the integral in the top rectangle of the striking set in Figure 1: Sam always strikes if $z_n > \zeta(c)$ — for if he enters stage n, then $\chi_n + \zeta(c) > \Omega_{n-1}$ by (5a) and (5b) — and the striking event $w_n \geq \max\{\bar{w}_{n+1}, \Omega_{n-1}\}$ holds (see Lemma 1). We prove that the second term in (25) is the conditional exercise chance if $z_n \leq \zeta(c)$.

Proof of Step 1: By (6) and (24), rewrite the conditional chance $\mathcal{E}_n \equiv e_n/\sigma_n$ as

$$\mathcal{E}_n = \frac{\int_u^\infty \delta(w^*-\zeta(c), c)^{n-1}G(w^*-\zeta(c))^{N-n}\phi(w^*)dw^*}{\int_{u-\zeta(c)}^\infty \delta(\chi, c)^{n-1}G(\chi)^{N-n}g(\chi)d\chi} = \frac{\int_{u-\zeta(c)}^\infty \frac{\phi(\chi+\zeta(c))}{g(\chi)}\eta(\chi)d\chi}{\int_{u-\zeta(c)}^\infty \eta(\chi)d\chi} \tag{26}$$

writing $\chi = w^* - \zeta(c)$, recalling $\eta(\chi) = \eta(\chi, c, n, N) \equiv \delta(\chi, c)^{n-1}G(\chi)^{N-n}g(\chi)$ by (15).

Since $\eta(\chi)$ is the probability density in (16) of $\mathcal{X}_n$ in the operator $E_{\mathcal{X}_n}$, (23) yields (25), as

$$\mathcal{E}_n = E_{\mathcal{X}_n}\left[\frac{\phi(\mathcal{X}_n + \zeta(c))}{g(\mathcal{X}_n)}\right] = E_{\mathcal{X}_n}\left[\frac{\int_{-\infty}^\infty h(\zeta(c)-s)g(\mathcal{X}_n + \max(s,0))ds}{g(\mathcal{X}_n)}\right].$$

Finally, (25) rises in n, as $\mathcal{X}_n$ stochastically falls in n by Lemma 5, and $\frac{g(s+x)}{g(x)}$ falls in x.□

Step 2 (Striking Chance Formula) *The conditional striking chance $\mathcal{K}_n$ equals*

$$\mathcal{K}_n = 1 - H(\zeta(c)) + E_{\mathcal{X}_n}\left(\int_0^\infty h\left(\zeta(c) - s\right) \frac{g(s+\mathcal{X}_n)}{g(\mathcal{X}_n)} \left[\frac{\int_s^\infty H(\zeta(c)-t)g(t+\mathcal{X}_n)dt}{\int_0^\infty H(\zeta(c)-t)g(t+\mathcal{X}_n)dt} \right]^{n-1} ds \right).$$

Proof: By Claim 3, Sam strikes if (21) holds and $\mathcal{X}' > \chi$ for $n-1$ options $(\mathcal{X}', \mathcal{Z}')$. The interim density $\phi_I(w^*)$ below modifies the *ex ante* density $\phi(w^*)$ in (23), conditioning on $\mathcal{X}' > \chi = w^* - \zeta(c) + \max\{0, s\}$ for $n-1$ options $(\mathcal{X}', \mathcal{Z}')$ — and so we divide by $P(\mathcal{X}' + \zeta(c) > w^* > \mathcal{X}' + \mathcal{Z}')$ for each of these $n-1$ options:

$$\phi_I(w^*) = \int_{-\infty}^\infty h(\zeta(c)-s) g(w^* - \zeta(c) + \max\{s, 0\}) \left[\frac{\int_{\max\{0,s\}}^\infty H(\zeta(c)-t)g(t+w^*-\zeta(c))dt}{\int_0^\infty H(\zeta(c)-t)g(t+w^*-\zeta(c))dt} \right]^{n-1} ds.$$

Sam strikes at stage n with *ex ante* chance $k_n = E_{\phi_I}[\Lambda_n(\mathcal{W}^*)] = \int_u^\infty \Lambda_n(w^*)\phi_I(w^*)dw^*$, recalling (22). By the logic deriving (25), as seen in (26), the conditional striking chance is

$$\mathcal{K}_n \equiv \frac{k_n}{\sigma_n} = \frac{\int_{u-\zeta(c)}^\infty [\phi_I(\chi + \zeta(c))/g(\chi)]\eta(\chi)d\chi}{\int_{u-\zeta(c)}^\infty \eta(\chi)d\chi} = E_{\mathcal{X}_n}\left[\frac{\phi_I(\mathcal{X}_n + \zeta(c))}{g(\mathcal{X}_n)} \right].$$

We can rewrite this expression as the desired formula by the proof logic in Step 1. □.

Step 3 (Recall Chance Formula) *The recall chance is $\mathcal{R}_n = E_{\mathcal{X}_n}[B(\mathcal{X}_n, n)]$, where*

$$B(\chi, n) \equiv \int_0^\infty h\left(\zeta(c) - s\right) \frac{g(s+\chi)}{g(\chi)} \left(1 - \left[\frac{\int_s^\infty H(\zeta(c)-t)g(t+\chi)dt}{\int_0^\infty H(\zeta(c)-t)g(t+\chi)dt} \right]^{n-1} \right) ds. \quad (27)$$

Proof: Formula (27) follows at once from Steps 1 and 2 and $\mathcal{R}_n = \mathcal{E}_n - \mathcal{K}_n$. □

Finally, we argue that $\mathcal{R}_n$ increases in n. This is subtle, for while $B(\chi, n)$ in (27) increases in n, the parenthetical factor in (27) increases in χ, and $\mathcal{X}_n$ falls stochastically in n. But Step 4 implies that $B(\chi, n)$ falls in χ, and so $\mathcal{R}_n = E_{\mathcal{X}_n}[B(\mathcal{X}_n, n)]$ rises in n.

Step 4 (Recall Chance) *$B(\chi, n)$ weakly falls in χ, and therefore $\mathcal{R}_n$ increases in n.*

Proof: Since Sam strikes (χ, z) whenever $z \geq \zeta(c)$, if he recalls (χ, z) then he once passed it over, and so $z < \zeta(c)$; thus, $w^*(\chi, z) = \chi + z < \chi + \zeta(c) = \bar{w}$. Also if Sam recalls (χ, z) at stage n, then he must have explored (χ, z), and hence $\chi > \mathcal{X}_n$. Given $w^*(\chi, z) = w \equiv \chi + z$ and $\chi > \mathcal{X}_n$, event (21b) is equivalent to the intersection of the next two events:

$$\mathcal{X}_n + \zeta(c) > w > \mathcal{X}_n + \mathcal{Z}_n \text{ and } \chi > \mathcal{X}_n, \quad (21b')$$

$$\mathcal{X}' > \mathcal{X}_n \text{ and } w > \mathcal{X}' + \mathcal{Z}' \text{ for } n-2 \text{ prior options } (\mathcal{X}', \mathcal{Z}'). \tag{21b''}$$

By (21b′), $(\mathcal{X}_n, \mathcal{Z}_n)$ obeys both inequalities in (21b) and (x, z) ranks before $(\mathcal{X}_n, \mathcal{Z}_n)$. By (21b″), $n-2$ other options satisfy (21b), also ranking before $(\mathcal{X}_n, \mathcal{Z}_n)$. By Claim 3, the *ex ante* chance of recall r_n is the *ex ante* chance of (21b′), (21b″), (21a) and (21c).

To compute r_n, let $\Upsilon(w)$ be the probability density of (i) the interim variable $\mathcal{W} \equiv \mathcal{X} + \mathcal{Z} = w$ for the target option, (ii) a given option $(\mathcal{X}_n, \mathcal{Z}_n)$ obeying (21b′) and (iii) $n-2$ options $(\mathcal{X}', \mathcal{Z}')$ obeying (21b″). By Claim 3, (21a), and (21c), the stage-n recall chance is

$$r_n = N(N-1)\binom{N-2}{N-n} \int_u^\infty \Upsilon(w) G(w - \zeta(c))^{N-n} dw \tag{29}$$

The coefficient is the number of ways to choose the target option (x, z), the last explored option $(\mathcal{X}_n, \mathcal{Z}_n)$, and the $n-2$ prior options $(\mathcal{X}', \mathcal{Z}')$, and $N-n$ later options $(\mathcal{X}'', \mathcal{Z}'')$.

First, the density of $w \equiv x + z$ is $\int_s^\infty h(\zeta(c) - t)\, g(t + w - \zeta(c)) dt$, where $x_n = s - \zeta(c) + w$. Event $\mathcal{X}_n + \zeta(c) > w$ in (21b′) has probability density $g(s + w - \zeta(c))$ for $s > 0$, and given x_n, the second inequality event in (21b′) has chance $P(\mathcal{Z}_n < w - x_n) = H(\zeta(c) - s)$. Each of the $n-2$ events in (21b″) has chance $\int_s^\infty H(\zeta(c)-t) g(t+w-\zeta(c)) dt = \iota(s, w - \zeta(c), \zeta(c))$, if $\iota(s, x, \zeta(c)) \equiv \int_s^\infty H(\zeta(c) - t) g(t + x) dt$. By independence of events:

$$\Upsilon(w) \equiv \int_0^\infty H(\zeta(c) - s) g(s + w - \zeta(c)) \left[\int_s^\infty h(\zeta(c) - t) g(t + w - \zeta(c)) dt\right] \iota(s, w - \zeta(c), \zeta(c))^{n-2} ds.$$

Recalling (6) and (29), then (15) and (16), the stage-n conditional recall chance equals

$$\mathcal{R}_n = \frac{r_n}{\sigma_n} = \frac{(n-1)\int_{u-\zeta(c)}^\infty \Upsilon(x + \zeta(c)) G(x)^{N-n} dx}{\int_{u-\zeta(c)}^\infty g(x)\delta(x, c)^{n-1} G(x)^{N-n} dx} = E_{\mathcal{X}_n}\left[\frac{(n-1)\Upsilon(\mathcal{X}_n + \zeta(c))}{g(\mathcal{X}_n)\delta(\mathcal{X}_n, c)^{n-1}}\right] \tag{30}$$

where $x = w - \zeta(c)$. Since $\mathcal{R}_n = E_{\mathcal{X}_n}[B(\mathcal{X}_n, n)]$ by Claim 3, the $B(\mathcal{X}_n, n)$ formula (27) is the bracketed term in (30). Since $B(x, n) = (n-1)\Upsilon(x + \zeta(c))/[g(x)\delta(x, c)^{n-1}]$,

$$\frac{B(x, n)}{n-1} = \int_0^\infty \frac{H(\zeta(c) - s) g(s + x)}{g(x)} \left[\frac{\int_s^\infty h(\zeta(c) - t)\, g(t + x) dt}{\int_s^\infty H(\zeta(c) - t) g(t + x) dt}\right] \nu(s, x, \zeta(c))^{n-1} ds. \tag{31}$$

where $\nu(s, x, \zeta(c)) \equiv \iota(s, x, \zeta(c))/\delta(x, c)$.

We argue that $B(x, n)$ falls in x: First, $g(s + x)/g(x)$ falls in x, as g is log-concave. Second, given g and H log-concave, $\mathbb{I}_{t \geq s} H(\zeta(c) - t) g(t + x)$ is log-supermodular in $(s, \zeta(c), -x, t)$, the integral $\iota(s, x, \zeta(c)) \equiv \int_s^\infty H(\zeta(c) - t) g(t + x) dt$ is log-supermodular in $(\zeta(c), -\chi)$, by Karlin and Rinott (1980), and so log-submodular in $(\zeta(c), \chi)$. Hence,

$$\frac{\int_s^\infty h\left(\zeta(c)-t\right)g(t+\chi)dt}{\int_s^\infty H(\zeta(c)-t)g(t+\chi)dt} = \frac{\partial \log[\iota(s,\chi,\zeta)]}{\partial \zeta} \tag{32}$$

falls in χ. By log-concavity of g, the following also falls in χ:

$$\frac{\partial \log[\iota(s,\chi,\zeta(c))]}{\partial s} = \frac{-H(\zeta(c)-s)g(s+\chi)}{\int_s^\infty H(\zeta(c)-t)g(t+\chi)dt} = \frac{-H(\zeta(c)-s)}{\int_0^\infty H(\zeta(c)-r-s)\frac{g(r+s+\chi)}{g(s+\chi)}dr} \tag{33}$$

Thus, $\iota(s,\chi,\zeta(c))$ is log-submodular in (s,χ), and $\nu(s,\chi,\zeta(c)) \equiv \iota(s,\chi,\zeta(c))/\iota(0,\chi,\zeta(c))$ falls in χ. Since $g(s+\chi)/g(\chi)$, and (32) and (33) fall in χ, $B(\chi,n)$ in (31) falls in χ. □

C.3 Search Duration and Prize Dispersion Proofs

Theorems 3–5 do not require log-concavity of the G and H distributions.

(a) Proof of the Increased Search Duration Claim in Theorem 4

Index the distribution H_t so that $\mathcal{Z}_t$ experiences a mean-enhancing dispersion as t rises. So the quantile function steepens in t, or $H_t^{-1}(\bar{\alpha}) - H_t^{-1}(\alpha)$ rises in t if $\bar{\alpha} > \alpha$; if differentiable, $\partial H_t^{-1}(\alpha)/\partial\alpha$ rises in t. Let $\zeta_t(c)$ solve the analogous Bellman equation (2).

Claim 4 *For any $\Delta > 0$, $H_t(\zeta_t(c)-\Delta)$ increases in the dispersion index t.*

Proof: Changing variables from z to $\alpha = H_t(z-\Delta)$ in the Bellman equation (2), we get:

$$c = \int_{H_t(\zeta_t(c)-\Delta)}^1 (1 - H_t(H_t^{-1}(\alpha)+\Delta))\frac{\partial H_t^{-1}(\alpha)}{\partial \alpha}d\alpha \tag{34}$$

Put $A_t(\alpha,\Delta) \equiv H_t(H_t^{-1}(\alpha)+\Delta)$. Then $H_t^{-1}(A_t(\alpha,\Delta)) - H_t^{-1}(\alpha) \equiv \Delta$, and so $A_t(\alpha,\Delta)$ falls in t (more disperse). As t rises, the integrand of (34) rises, as does $H_t(\zeta_t(c)-\Delta)$.□

Claim 5 (Survival Chances) *σ_n rises in t.*

Proof: By (6), the survival chance σ_n rises in $\delta_t(\chi,c)$ and falls in $u-\zeta_t(c)$. By Claim 4, $H_t(\zeta_t(c)-s)$ rises in t. From (4), $\delta_t(\chi,c) = \int_0^\infty H_t(\zeta_t(c)-s)g(x+s)ds$ rises in t, by (4). Thus, σ_n increases if $u-\zeta_t(c)$ falls in t. Now, a mean-enhancing dispersion in $\mathcal{Z}_t$ is a mean-preserving dispersion of $\mathcal{Z}_t$ plus a positive constant. Also, $\zeta_t(c)$ increases in any MPS of $\mathcal{Z}_t$ by (2), and thus in a mean-preserving dispersion. Also, $\zeta_t(c)$ rises when $\mathcal{Z}_t$ shifts up by a positive constant, by (2). Then $\zeta_t(c)$ rises in t, and thus $u-\zeta_t(c)$ falls. □

(b) Proof of the Recall Moment Claim in Theorem 4

The chance ρ_n that the recall moment is at least $n \geq 1$ falls in n. By Lemma 1, Sam strikes or passes in stage n if the fallback is at most the stage $n+1$ cutoff $\mathcal{X}_{n+1}+\zeta(c)$. So

$$\begin{aligned}\rho_n &= P(\max_{j\leq n-1}\{u, \mathcal{X}_j + \mathcal{Z}_j\} < \mathcal{X}_{n+1} + \zeta(c)). \qquad (35)\\ &= N\binom{N-1}{n-1}\int_{u-\zeta(c)}^{\infty}\int_{x_{n+1}}^{\infty}\left[\int_{x_n}^{\infty} H(x_{n+1}+\zeta(c)-x)dG(x)\right]^{n-1} dG(x_n)dG(x_{n+1})^{N-n}\end{aligned}$$

By Claim 4, $H(x_{n+1}+\zeta(c)-x)$ increases as $\mathcal{Z}$ grows more dispersed, as $x \geq x_n > x_{n+1}$. Also, as $\zeta(c)$ increases for any mean-preserving dispersion in $\mathcal{Z}$ (footnote 21), the integration domain expands. Hence, ρ_n rises if $\mathcal{Z}$ experiences a mean-enhancing dispersion; thus, the recall moment stochastically rises if $\mathcal{Z}$ incurs a mean-preserving dispersion.

Claim 6 *The recall moment stochastically increases in the number of options N.*

Proof: By the Markov property of order statistics, the joint distribution of the n top known factors is that of n i.i.d. draws $\mathcal{X}$ from G, given $\mathcal{X} > x_{n+1}$ (proof of Theorem 2), namely, with cdf $\tilde{G}(x) = G(x)/[1-G(x_{n+1})]$ on $[x_{n+1}, \infty)$. Since $1-G$ is log-concave, $\tilde{G}(x)$ grows less disperse as x_{n+1} increases (Theorem 3.B.19 in Shaked and Shanthikumar (2007)). So the gap between draws from $\tilde{G}$ and x_{n+1} stochastically shrinks. As N increases, $u-\mathcal{X}_{n+1}$ and $\mathcal{X}_j-\mathcal{X}_{n+1}$ stochastically fall, and so ρ_n rises, since (35) implies:

$$\rho_n = P(\max_{j\leq n-1}\{u-\mathcal{X}_{n+1}, \mathcal{X}_j-\mathcal{X}_{n+1}+\mathcal{Z}_j\} < \zeta(c)) \qquad \square$$

(c) Search Duration and Participation: Proof of Theorem 5

Write $\mathcal{X}_j - \mathcal{X}_n = \Sigma_{k=j}^{n-1}(\mathcal{X}_k - \mathcal{X}_{k+1}) = \Sigma_{k=j}^{n-1}\Delta_k$, for gaps $\Delta_k \equiv \mathcal{X}_k - \mathcal{X}_{k+1} \geq 0$. Then

$$\sigma_n = P(\{\mathcal{X}_n + \zeta(c) > \Omega_{n-1}) = P\left(\{\mathcal{X}_n+\zeta(c) > u\} \cap_{j<n} \left\{\zeta(c) - \Sigma_{k=j}^{n-1}\Delta_k \geq \mathcal{Z}_j\right\}\right)$$

is the chance of event (5). Using the joint distribution ψ of $\vec{\Delta}_n = (\Delta_1, \ldots, \Delta_{n-1})$ and $\mathcal{X}_n$:

$$\sigma_n = \int_{x_n\in\mathbb{R}, \vec{\Delta}_n \in \mathbb{R}_+^{n-1}} \mathbb{I}_{\{x_n+\zeta(c)\geq u\}} \prod_{j=1}^{n-1} H(\zeta(c) - \Sigma_{k=j}^{n-1}\Delta_k) d\psi(\vec{\Delta}_n, x_n). \qquad (36)$$

With no outside option ($u=-\infty$), the indicator $\mathbb{I}=1$ in (36). As $\mathcal{X}$ grows less dispersive, quantiles grow closer by (8), and $\vec{\Delta}_n \equiv \{\mathcal{X}_1-\mathcal{X}_2, \ldots, \mathcal{X}_j-\mathcal{X}_{j+1}, \ldots, \mathcal{X}_{n-1}-\mathcal{X}_n\}$

falls stochastically, namely, all gaps $\mathcal{X}_j - \mathcal{X}_{j+1}$ stochastically fall for $j = 1, \ldots, n-1$, and thus σ_n rises. Next suppose, $u > -\infty$. If $\mathcal{X}$ stochastically rises, then so does the order statistic $\mathcal{X}_n$ in (36). Since $\mathbb{I}_{\{\mathcal{X}_n \geq u - \zeta(c)\}}$ rises in $\mathcal{X}_n$, so does σ_n. □

(d) Search with Known Factors is Shorter if $\zeta(c) > u$: Proof of Corollary 1

Let $\tau(u, \mathcal{X})$ be the search duration. We argue that $\tau(u, \mathcal{X}) < \tau(u, 0)$ iff $\zeta(c) > u$.

Assume $\zeta(c) > u$. First, $\tau(-\infty, \mathcal{X}) < \tau(-\infty, 0)$ for nondegenerate $\mathcal{X}$ (proof of Theorem 5). As noted after (6), search duration falls in u, for non-degenerate $\mathcal{X}$, i.e. $\tau(u, \mathcal{X}) < \tau(-\infty, \mathcal{X})$ for all $u > -\infty$. But when $\mathcal{X} = 0$, Sam never stops if $u < \zeta(c)$, and so search duration is constant in u, or $\tau(-\infty, 0) = \tau(u, 0)$. So $\tau(u, \mathcal{X}) < \tau(-\infty, \mathcal{X}) < \tau(-\infty, 0) = \tau(u, 0)$. Finally, if $\zeta(c) \leq u$ then search duration is $\tau(u, \mathcal{X}) > \tau(u, 0) = 0$. □

(e) Example: Mean-Preserving Spread (MPS) Can Reduce Search Duration

We argue that search duration increases in γ iff $\gamma < \gamma^*(c)$, and also iff $c > c^*(\gamma)$.

Assume $-\mathcal{Z}$ is Pareto with shape parameter $\gamma > 1$ and $\mathcal{Z}$ has support $(-\infty, \bar{z}]$ with $\bar{z} < 0$. By the definition of the Pareto distribution the cdf of $\mathcal{Z}$ is $H(z) = (\bar{z}/z)^\gamma$ and its mean is given by $E[\mathcal{Z}] = \gamma\bar{z}/(\gamma - 1)$. We restrict $E[\mathcal{Z}] = -1$ by setting $\bar{z} = -1 + 1/\gamma$. Then the cdf becomes $H(z) = [(-1 + 1/\gamma)/z]^\gamma$ and its support is $(-\infty, -1 + 1/\gamma]$. The density h and cdf H are both log-convex in this example, inspired by footnote 17.

Near $\gamma \approx 1$, if γ falls, then the next Claim 7 implies that $\mathcal{Z}$ experiences a MPS, while the subsequent Claim 8 asserts that the stopping chance rises (i.e. search duration falls).

Claim 7 *If γ falls, then $\mathcal{Z}$ incurs a MPS, but $\mathcal{Z}$ does not grow more disperse if $\gamma < 2$.*

Proof: Let $\gamma_B > \gamma_A > 1$. For $b \in (0, 1)$ and $a = A, B$, the quantile function is $H_a^{-1}(b) = -b^{-1/\gamma_a}(1 - \gamma_a)/\gamma_a$. Since the means of H_A and H_B are -1 by construction, if $H_A^{-1}(b) - H_B^{-1}(b)$ is single-crossing, then H_A is a MPS of H_B, by Diamond and Stiglitz (1974). Since $H_A^{-1}(b) - H_B^{-1}(b) = H_A^{-1}(b)\left[1 - H_B^{-1}(b)/H_A^{-1}(b)\right]$, and $H_A^{-1}(b) < 0$, it suffices to show

$$\frac{H_B^{-1}(b)}{H_A^{-1}(b)} = \frac{\gamma_A(\gamma_B - 1)}{\gamma_B(\gamma_A - 1)} b^{\frac{1}{\gamma_A} - \frac{1}{\gamma_B}}$$

rises in b — which holds, as $\gamma_B > \gamma_A > 1$. So H_A is a mean-preserving spread of H_B.

Next consider the change in the slope of the quantile function with respect to γ

$$\frac{d}{d\gamma}\left[\frac{dH^{-1}(b)}{db}\right] = \frac{d}{d\gamma}\left[\frac{\gamma - 1}{\gamma^2 b^{1/\gamma+1}}\right] = \frac{\gamma - 1}{\gamma^2 b^{1/\gamma+1}}\left[\frac{2 - \gamma}{\gamma(\gamma - 1)} + \frac{1}{\gamma^2}\log(b)\right] \tag{37}$$

Since $\log(b) \in (-\infty, 0)$, for $\gamma \in (1,2)$, expression (37) is positive iff b is large enough. So $H^{-1}(b)$ does not flatten for all $b \in (0,1)$ as γ increases: Dispersion need not rise in γ. □

Claim 8 (Rising γ) *If $c > 0$, then $1 - H(\zeta(c))$ falls in γ iff $\gamma < \gamma^*$, for some $\gamma^* > 1$.*

Proof: If $H(z) = [(-1 + 1/\gamma)/z]^\gamma$, the Bellman equation (2) becomes

$$(c+1)\gamma H(\zeta)^{1/\gamma} - \gamma + 1 - H(\zeta) = 0. \tag{38}$$

A unique solution, say $H_\gamma(\zeta)$ exists, because the LHS increases in $H(\zeta)$, is negative if $H(\zeta) = 0$, and positive if $H(\zeta) = 1$. Differentiating (38) in γ, we have $dH_\gamma(\zeta)/d\gamma \geq 0$ iff

$$H_\gamma(\zeta)^{1/\gamma}[1 - \log(H_\gamma(\zeta)^{1/\gamma})] \leq \frac{1}{c+1}. \tag{39}$$

We claim inequality (39) is strict iff $\gamma < \gamma^*$, for $\gamma^* > 1$. Now, (39) is strict for γ near 1, since $x[1 - \log(x)] \downarrow 0$ as $x \downarrow 0$, and $H(\zeta) \downarrow 0$ as $\gamma \downarrow 1$ by (38). Once $\gamma > 1$, if (39) binds at some γ^*, then (39) is strict and $dH_\gamma(\zeta)/d\gamma > 0$ for $\gamma < \gamma^*$, and $dH_\gamma(\zeta)/d\gamma = 0$ at $\gamma = \gamma^*$. But then $H_\gamma(\zeta)^{1/\gamma}$ increases in γ at $\gamma = \gamma^*$, and so (39) rises in γ at $\gamma = \gamma^*$. Hence, (39) fails for all $\gamma > \gamma^*$, proving that $dH_\gamma(\zeta)/d\gamma > 0$ iff $\gamma < \gamma^*$. □

C.4 Stationary Benchmark Proofs

(a) Equivalent Thin Tail Characterizations

By log-concavity, $\ell = \lim_{\chi \to F^{-1}(1)} f(\chi)/[1 - F(\chi)]$ exists. If F has a thin tail, then $\ell = \infty$.

Claim 9 *If $F^{-1}(1) = \infty$, then $\lim_{\chi \to F^{-1}(1)} f(s+\chi)/f(\chi) = e^{-s\ell}$ for all $s > 0$.*

Proof: As $F^{-1}(1) = \infty$, for all $s > 0$:

$$\lim_{\chi\to\infty} \log\left(\frac{1 - F(s+\chi)}{1 - F(\chi)}\right) = \lim_{\chi\to\infty} \int_0^s \frac{-f(r+\chi)}{1 - F(r+\chi)} dr = \int_0^s \lim_{\chi\to\infty} \frac{-f(r+\chi)}{1 - F(r+\chi)} dr = -s\ell,$$

exchanging integration and limits by the Monotone Convergence Theorem, for $f/(1-F)$ is monotone if f is log-concave. For all $s > 0$, l'Hôpital's rule and exponentiation imply:

$$\lim_{\chi\to\infty} \frac{f(s+\chi)}{f(\chi)} = \lim_{\chi\to\infty} \exp\left[\log\left(\frac{1 - F(s+\chi)}{1 - F(\chi)}\right)\right] = \exp\left[\lim_{\chi\to\infty} \log\left(\frac{1 - F(s+\chi)}{1 - F(\chi)}\right)\right] = e^{-s\ell}$$

Claim 10 *If $F^{-1}(1) = \infty$, then F has a thin tail iff $\lim_{\chi \to F^{-1}(1)} \frac{f(s+\chi)}{f(\chi)} = 0$ for all $s > 0$.*

Proof: Given a thin tail, $\ell=\infty$ and $f(s{+}\chi)/f(\chi)\to 0$ for $s>0$ by Claim 9. Conversely, if $\lim_{\chi\to\infty} f(s+\chi)/f(\chi)=0\ \forall s>0$, then $\lim_{\chi\to F^{-1}(1)} f(\chi)/[1-F(\chi)]$ equals

$$\lim_{\chi\to\infty}\left(\int_0^\infty \frac{f(s+\chi)}{f(\chi)}ds\right)^{-1}=\left(\lim_{\chi\to\infty}\int_0^\infty \frac{f(s+\chi)}{f(\chi)}ds\right)^{-1}=\left(\int_0^\infty \lim_{\chi\to\infty}\frac{f(s+\chi)}{f(\chi)}ds\right)^{-1}=\infty.$$

by continuity and the Monotone Convergence Theorem. Hence, F has a thin tail. □

(c) Increasing the Total Number of Options: Proof of Theorems 6 and 7

Index the striking, recall and quitting hazard rates by the number of options N. We argue that $\mathcal{K}_n^N$, $\mathcal{R}_n^N$, and $\mathcal{Q}_n^N$ weakly fall in N, and so the limits $\mathcal{K}_n^\infty$, $\mathcal{R}_n^\infty$, and $\mathcal{Q}_n^\infty$ exist.

Claim 11 *The known factor $\mathcal{X}_n^N$ conditional on entering stage n converges to $G^{-1}(1)$ in probability as $N\to\infty$, namely, $\lim_{N\to\infty} P(\mathcal{X}_n^N\le a|\text{enter stage } n){=}0$ for all $a<G^{-1}(1)$.*

Proof: By (15) and (16), the cdf of $\mathcal{X}_n$ is

$$P(\mathcal{X}_n^N\le a|\text{enter stage } n)=\frac{\int_{u-\zeta(c)}^{a}\delta(\chi,c)^{n-1}[G(\chi)/G(a)]^{N-n}g(\chi)d\chi}{\int_{u-\zeta(c)}^{\infty}\delta(\chi,c)^{n-1}[G(\chi)/G(a)]^{N-n}g(\chi)d\chi}.$$

As $N\to\infty$, the numerator vanishes as $G(\chi)/G(a)<1$ for all $\chi\in[u-\zeta(c),a)$, and the denominator explodes, as $G(\chi)/G(a)>1$ for $\chi\in(a,\infty)$ and the density $g(\chi)$ has positive mass on (a,∞) if $a{<}G^{-1}(1)$. Thus, $\lim_{N\to\infty} P(\mathcal{X}_n^N\le a){=}0\ \forall a{<}G^{-1}(1)$. □

Claim 12 (Striking Chance) *Fix n. The conditional striking chance $\mathcal{K}_n^N$ falls in N. The limit chance $\mathcal{K}_n^\infty=1-H(\zeta(c))$ if G has a thin tail, and $\mathcal{K}_n^\infty>1-H(\zeta(c))$ if not.*

Proof: Write $\mathcal{K}_n^N{=}1-H(\zeta(c)){+}E_{\mathcal{X}_n^N}[\Gamma(s,\mathcal{X}_n^N)]$, where $\Gamma(s,\mathcal{X}_n^N)$ is the bracketed term in Step 2. As $\iota(s,\chi,\zeta(c))\equiv\int_s^\infty H(\zeta(c)-t)g(t+\chi)dt$ is log-submodular in (s,χ) by (33) and log-concavity of g, and $g(s+\chi)/g(\chi)$ weakly falls in χ by log-concavity, we have ($\diamondsuit$): $\Gamma(s,\mathcal{X}_n^N)$ *falls in* $\mathcal{X}_n^N$. As $\mathcal{X}_n^N$ stochastically rises in N by Lemma 5, $\mathcal{K}_n^N$ falls in N.

The n^{th} known factor $\mathcal{X}_n^N\to G^{-1}(1)=\infty$ in probability by Lemma 11, as $N\to\infty$. If G has a thin tail, then $\lim_{\chi\to\infty} g(s+\chi)/g(\chi)=0$ for $s>0$, by Claim 10, and so $g(s+\mathcal{X}_n)/g(\mathcal{X}_n^N){\downarrow}0$ as $N{\to}\infty$. In this case, Step 2 implies $\lim_{N\to\infty}\mathcal{K}_n^N=1-H(\zeta(c))$.

Assume G lacks a thin tail. As $\lim_{\chi\to\infty} g(t{+}\chi)/g(\chi){=}e^{-t\ell}$ for $\ell\in(0,\infty)$ by Claim 9,

$$\Gamma(s,\chi)\equiv\frac{\int_s^\infty H(\zeta(c)-t)g(t+\chi)/g(\chi)dt}{\int_0^\infty H(\zeta(c)-t)g(t+\chi)/g(\chi)dt}\to\frac{\int_s^\infty H(\zeta(c)-t)e^{-\ell t}dt}{\int_0^\infty H(\zeta(c)-t)e^{-\ell t}dt}{>}0 \text{ as } \chi\to\infty. \quad (40)$$

By (◇), $\Gamma(s,\chi)$ falls in χ, tending to $\lim_{\chi\to\infty}\Gamma(s,\chi) > 0$ by (40). Since $\mathcal{X}_n^N$ increases stochastically in N, by Lemma 5, we have $\lim_{N\to\infty} E_{\mathcal{X}_n^N}[\Gamma(s,\mathcal{X}_n^N)] > 0$ by the Continuous Mapping Theorem, and therefore, $\mathcal{K}_n^\infty > 1 - H(\zeta(c))$. □

Claim 13 (Recall) *$\mathcal{R}_n^N$ falls in N and the limit $\mathcal{R}_n^\infty = 0$ iff G has a thin tail. The limit $\mathcal{E}_n^\infty \equiv \mathcal{R}_n^\infty + \mathcal{K}_n^\infty$ is $1 - H(\zeta(c))$ if G has a thin tail, and $\mathcal{E}_n^\infty \in (1 - H(\zeta(c)), 1)$ if not.*

Proof: Since $B(\chi,n)$ falls in χ by Step 4, $\mathcal{R}_n^N = E_{\mathcal{X}_n}[B(\mathcal{X}_n,n)]$ falls in N, as $\mathcal{X}_n$ increases stochastically in N (Lemma 5). So $\mathcal{R}_n^\infty = \lim_{N\to\infty} E_{\mathcal{X}_n}[B(\mathcal{X}_n,n)] = \lim_{\chi\to G^{-1}(1)} B(\chi,n)$ by the Continuous Mapping Theorem, as $\mathcal{X}_n \to G^{-1}(1)$ in probability if $N\uparrow\infty$ (Lemma 11).

If G has a thin tail, and $s>0$, then $g(s+\chi)/g(\chi)\downarrow 0$ as $\chi\to G^{-1}(1)=\infty$, by Claim 10; so $B(\chi,n)\downarrow 0$ by (27). With no thin tail, $(1-\Gamma(s,\chi)^{n-1})$ in (27) is boundedly positive as $\chi\to G^{-1}(1)$ (by (40)); so $\lim_{\chi\to G^{-1}(1)} B(\chi,n) > 0$ by (27), i.e. $\mathcal{R}_n^\infty = 0$ iff G has a thin tail.

By Claim 1, $\mathcal{E}_n^\infty = 1 - H(\zeta(c))$ if G has a thin tail; if not, $\mathcal{E}_n^\infty \in (1 - H(\zeta(c)), 1)$. □

Claim 14 (Quitting) *Regardless of G, $\mathcal{Q}_n^N$ falls in N, and tends to the limit $\mathcal{Q}_n^\infty = 0$.*

Proof: Expanding $\mathcal{Q}_n \equiv q_n/\sigma_n$ using (13) and (6), respectively:

$$\mathcal{Q}_n^N = \frac{\delta(u-\zeta(c),c)^n G(u-\zeta(c))^{N-n}}{n\int_{u-\zeta(c)}^\infty \delta(\chi,c)^{n-1} G(\chi)^{N-n} g(\chi)d\chi}.$$

Easily, $\mathcal{Q}_n^N$ falls in N, since $G(\chi)/G(u-\zeta(c)) > 1$ except at $\chi = u-\zeta(c)$, and thus $[G(\chi)/G(u-\zeta(c))]^{N-n}$ is monotone in N. By the monotone convergence theorem, we can swap the (infinite) limit as $N\to\infty$ and integration: $\lim_{N\to\infty} \mathcal{Q}_n^N = \mathcal{Q}_n^\infty = 0$. □

Claim 15 (Duration) *Search duration rises in N.*

Proof: Since $\mathcal{Q}_n$, $\mathcal{R}_n$ and $\mathcal{K}_n$ fall in N, search duration τ increases in N. For the striking hazard rate $\mathcal{S}_k \equiv 1 - \sigma_{k+1}/\sigma_k$ yields (by a telescoping product) the survival chance formula $\sigma_k = \sigma_1 \Pi_{j=1}^{k-1}(1-\mathcal{S}_j)$. As N rises, so does this product, as $\sigma_1 = P(\mathcal{X}_1 > u-\zeta(c))$ rises by Lemma 5, and every $\mathcal{S}_j \equiv \mathcal{Q}_j + \mathcal{E}_j$ falls. Duration $\tau \equiv \sum_{k=1}^N \sigma_k$ thus rises in N. □

Claim 16 (Limit Recall Chance) *Absent a thin tail, $\mathcal{R}_n^\infty$ rises in dispersion of $\mathcal{X}$.*

Proof: By Claim (31) and $\nu(s,\chi,\zeta(c)) \equiv \int_s^\infty H(\zeta(c)-t)g(t+\chi)dt / \int_0^\infty H(\zeta(c)-t)g(t+\chi)dt$,

$$\frac{B(\chi,n)}{n-1} = \int_0^\infty \frac{H(\zeta(c)-s)g(s+\chi)}{g(\chi)} \left[\frac{\int_s^\infty h(\zeta(c)-t)\frac{g(t+\chi)}{g(\chi)}dt}{\int_s^\infty H(\zeta(c)-t)\frac{g(t+\chi)}{g(\chi)}dt}\right]\left[\frac{\int_s^\infty H(\zeta(c)-t)\frac{g(t+\chi)}{g(\chi)}dt}{\int_0^\infty H(\zeta(c)-t)\frac{g(t+\chi)}{g(\chi)}dt}\right]^{n-1} ds.$$

By Claim 9, if G lacks a thin tail, then $\lim_{\chi\to G^{-1}(1)} g(s{+}\chi)/g(\chi)=e^{-s\ell}$ for all $s>0$, where $\ell\equiv\lim_{a\to 1} g(G^{-1}(a))/(1{-}a)$. Since $\mathcal{R}_n^\infty\equiv\lim_{N\to\infty} E_{\mathcal{X}_n^N}[B(\mathcal{X}_n^N,n)]=\lim_{\chi\to G^{-1}(1)} B(\chi,n)$:

$$\mathcal{R}_n^\infty=(1-n)\int_0^\infty H(\zeta(c)-s)e^{-s\ell}\left[\frac{\int_s^\infty h\left(\zeta(c)-t\right)e^{-t\ell}dt}{\int_s^\infty H(\zeta(c)-t)e^{-t\ell}dt}\right]\left[\frac{\int_s^\infty H(\zeta(c)-t)e^{-t\ell}dt}{\int_0^\infty H(\zeta(c)-t)e^{-t\ell}dt}\right]^{n-1}ds.$$

The limit $\ell\equiv\lim_{a\to1} g(G^{-1}(a))/(1-a)$ falls in the dispersion of $\mathcal{X}$. As in the proof of Step 4, $\int_s^\infty H(\zeta(c)-t)e^{-t\ell}dt$ is log-submodular in (ℓ,ζ) and in (ℓ,s), by log-concavity of H. So $\mathcal{R}_n^\infty$ increases in the dispersion of $\mathcal{X}$, as each bracketed factor above falls in ℓ. □

Claim 17 *The recall moment rises with a mean-preserving dispersion increase of $\mathcal{Z}$.*

Proof: By Claim 4, $H(x_{n+1}+\zeta(c)-x)$ increases in the dispersion of $\mathcal{Z}$. Also, the optionality value $\zeta(c)$ increases as $\mathcal{Z}$ incurs a mean-preserving dispersion, reducing the lower support $u-\zeta(c)$ of the integral in (35). Altogether, ρ_n increases. □

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