

Gradual Bargaining in Decentralized Asset Markets*

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Abstract

We introduce a new approach to bargaining, with strategic and axiomatic foundations, into models of decentralized asset markets. According to this approach, which encompasses the Nash (1950) solution as a special case, bilateral negotiations follow an agenda that partitions assets into bundles to be sold sequentially. We construct two alternating-offer games consistent with this approach and characterize their subgame perfect equilibria. We show the revenue of the asset owner is maximized when assets are sold one infinitesimal unit at a time. In a general equilibrium model with endogenous asset holdings, gradual bargaining reduces asset misallocation and prevents market breakdowns.

JEL Classification: D83

Keywords: decentralized asset markets, bargaining with an agenda, Nash program.

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1 Introduction

Modern monetary theory and financial economics formalize asset trades in the context of decentralized markets with explicit game-theoretic foundations (e.g., Duffie et al., 2005; Lagos and Wright, 2005). These models replace the elusive Walrasian auctioneer by a market structure with two core components: a technology to form pairwise meetings and a strategic or axiomatic mechanism to determine prices and trade sizes. This paper focuses on the latter: the negotiation of asset prices and trade sizes in pairwise meetings.

Going back to Diamond (1982), the search-theoretic literature has placed stark restrictions on individual asset inventories, typically $a \in \{0, 1\}$. As a result, in versions of the model with bargaining (e.g., Shi, 1995; Trejos and Wright, 1995; Duffie et al., 2005), the only item to negotiate in pairwise meetings — the *agenda* of the negotiation — is the price of an indivisible asset in terms of a divisible commodity.¹ Recent incarnations of the model (surveyed in Lagos et al., 2017) allow for unrestricted portfolios of divisible assets, $\mathbf{a} \in \mathbb{R}_+^J$ with $J \in \mathbb{N}$. A key conceptual difference when $\mathbf{a} \in \mathbb{R}_+^J$ is that the agenda of the negotiation is not unique. Any ordered partition of $\mathbf{a} \in \mathbb{R}_+^J$ constitutes an agenda, where the elements of this partition correspond to items to be negotiated sequentially. For instance, agents can sell their whole portfolio at once, as a large block, or they can partition their portfolio into bundles of varying compositions and sizes to be added to the negotiation table one after another.

The possibility of negotiating asset sales according to different agendas raises several questions regarding trading strategies and price formation in decentralized asset markets. Do agendas matter for asset prices and trade sizes when agents have perfect foresight and information is complete? What is the optimal strategy of the asset owner to partition his portfolio, e.g., should the portfolio be negotiated as a whole or divided into smaller bundles? What is the relation between the bargaining problem with an agenda and the bargaining solution of Nash (1950)?

Our contribution is to introduce a new and generalized approach to bargaining over

¹A thorough treatment of the axiomatic and strategic solutions for such bargaining problems is provided by Osborne and Rubinstein (1990). In Osborne and Rubinstein (1990) agents trade an indivisible consumption good and pay with transferable utility. The interpretation is reversed in Shi (1995) and Trejos and Wright (1995) where the indivisible good is fiat money and agents negotiate over a divisible consumption good. In Duffie et al. (2005) the indivisible good is a consol and agents pay with transferable utility.

portfolios of assets in models of decentralized asset market with the notion of agenda at the forefront, under both strategic and axiomatic foundations. The paper is composed of two parts, one about bargaining and one about markets. The first part provides a detailed description of bargaining games with an agenda and derives a series of methodological results that will be useful to incorporate these bargaining games into a general market structure. The second part focuses on the general equilibrium and derives some implications of the agenda of the negotiation for asset prices, allocations, and welfare.

We start with a simple agenda that partitions a portfolio of homogeneous assets into N bundles of equal sizes. This agenda is a natural extension of the negotiation in Shi (1995) and Trejos and Wright (1995), where the indivisible asset is now interpreted as a bundle of divisible assets. The extensive-form bargaining game, called the alternating-ultimatum-offer game, is composed of N rounds. In each round, one asset bundle is up for negotiation. One player makes an ultimatum offer, and the identity of the proposer alternates across rounds. Agents are forward-looking and can anticipate the outcomes of future rounds. In contrast to the Rubinstein game, our game is nonstationary, since the amount of assets left for negotiation decreases over time, and it admits a unique subgame-perfect equilibrium (SPE) characterized by a system of difference equations with initial condition allowing us to compute the terminal allocation in closed-form for all N .

The limit as N goes to infinity is called the *gradual solution*. It gives a simple and intuitive relationship between asset prices and trade sizes, and it has properties distinct from the Nash (1950) solution that make it tractable for general equilibrium analysis, including monotonicity and concavity of trade surpluses with respect to trade size. Moreover, it coincides with the axiomatic ordinal solution of O'Neill et al. (2004) where an agenda is defined as a collection of Pareto frontiers indexed by time.

In order to relate our approach to the Nash solution, commonly used in the asset market literature (e.g., Duffie et al., 2005; Lagos and Wright, 2005), we extend our N -round game by assuming that in each round agents play an alternating-offer game with risk of breakdown, as in Rubinstein (1982). The equilibrium allocation of this N -round game is obtained by applying the Nash solution consecutively N times, where the solution in one iteration becomes the disagreement point of the next iteration. We characterize the outcome in closed form for all N and show it coincides with the Nash solution and the gradual solution in the

two limiting cases $N = 1$ and $N = +\infty$, respectively. We endogenize the agenda of the negotiation by letting asset owners choose N to maximize their surplus from trade. The optimal choice is $N = +\infty$, i.e., it is optimal for the owner to add assets on the bargaining table gradually, one infinitesimal unit at a time.

The second part of the paper incorporates bargaining solutions with an agenda into a general equilibrium model of decentralized asset markets with endogenous portfolios along the lines of Lagos and Wright (2005) and Lagos and Zhang (2020). The equilibrium under Nash bargaining ($N = 1$) features asset misallocation: a fraction of the asset supply ends up being held by agents with no liquidity needs. In contrast, under gradual bargaining ($N = +\infty$), the first best is implemented as long as the asset supply is sufficiently abundant. In the case of fiat money, the optimal policy, the Friedman rule, generates the first best under gradual bargaining for all bargaining powers whereas it fails to do so under generalized Nash bargaining as long as producers have some bargaining power. Using the same calibrated parameter values as in Lagos and Wright (2005), going from $N = 1$ to $N = +\infty$ increases output and consumption at the optimal policy by 76%. Even a moderate increase from $N = 1$ to $N = 5$ raises output by 39%.

This finding is especially stark in a monetary version of the model where agents trade short-lived assets that they value according to linear preferences, e.g., as in the model of OTC market of Lagos and Zhang (2020). We allow agents to choose how much of their short-lived assets to bring into a match. Under Nash bargaining, the OTC market shuts down for all interest rates and the equilibrium achieves its worst allocation. This result is a direct consequence of the non-monotonicity of agents' surpluses with respect to the quantity of goods or assets that they bring to the negotiation table. Under gradual bargaining, the OTC market is active and the equilibrium achieves first best for all interest rates below a positive threshold.

Finally, we extend our environment to allow for any arbitrary number of assets. All assets, except fiat money, generate the same stream of dividends. The notion of agenda allows us to introduce a new asset characteristic – negotiability – defined as the inverse of the amount of time required for the sale of each unit of the asset to be finalized, e.g., each asset added to the negotiation table needs to be authenticated and ownership rights take

time to transfer.² Our model generates an endogenous pecking order: assets that are more negotiable are put on the negotiating table before the less negotiable ones. In equilibrium, the most negotiable assets have lower rates of return and higher velocities. Hence, our model explains rate-of-return differences of seemingly identical assets. As the time horizon of the negotiation becomes arbitrarily large, differences in rates of return vanish but differences in velocities persist. We discuss the potential of our model to address two puzzles in monetary theory, the rate-of-return dominance puzzle and the indeterminacy of the exchange between two fiat currencies.

Related literature

Models of decentralized markets adopting a strategic approach to the bargaining problem in pairwise meetings were pioneered by Rubinstein and Wolinsky (1985). Bargaining with an agenda composed of multiple issues was first studied by Fershtman (1990). The axiomatic formulation with a continuous agenda was developed by O'Neill et al. (2004). We provide both its first application in the context of decentralized asset market models and strategic foundations with two extensive-form games that admit as limiting outcomes the ordinal solution of O'Neill et al. (2004). Another important distinction relative to the work of O'Neill et al. is the fact that we specify the agenda in terms of the agents' initial endowments or asset holdings and not only in terms of utility space. Our approach allows us to identify agendas that are meaningful in the context of decentralized markets.

Our contribution on the strategic foundations was influenced by an earlier working paper by Wiener and Winter (1998, Section 8) where they assert that the relevant limits of three distinct bargaining games with alternating offers should generate the same outcome as the ordinal solution of O'Neill et al. (2004).³ We provide a complete and rigorous proof of this statement in the context of an over-the-counter bargaining game with forward-looking agents and liquidity constraints.

²The concept of negotiability dates back to the 17th century and referred to institutional arrangements aiming at enhancing liquidity by "centralizing all rights to the underlying asset in a single physical document, [...] reducing the costs a prospective purchaser incurs in acquiring [...] information about the asset" (Mann, 1996). The concept of blockchains - immutable, decentralized ledgers that can record ownership and transfer of intangible assets - can be seen as a digital incarnation of the original idea of negotiability.

³The relevant results from Wiener and Winter (1998) are contained in their Propositions 5 and 7. Because their Appendix 4 only contains sketches of proofs, it is unclear whether those results would apply to our setting with forward-looking agents and liquidity constraints. Hence, we set up precise extensive-form games and prove equivalence results in the context of our model.

Our second game based on a "repeated" Stahl-Rubinstein game is related to the Stole and Zwiebel (1996) game in the literature on intra-firm wage bargaining. See Brugemann et al. (2018) for a recent re-examination of this game. Some of the key differences are as follows. In the intra-firm bargaining literature workers sell an indivisible unit of labor, whereas in models of asset markets agents sell divisible assets. Moreover, we let agents choose both the quantity of assets to sell and the number of rounds of the negotiation. The extensive-form of the game is also different. In our game, if agents fail to reach an agreement in one round, they move to the next round, but the agreements of earlier rounds are preserved. In the Stole-Zwiebel game, all previous agreements are erased.

The "repeated" Rubinstein game is also used in Hu and Rocheteau (2020) to establish strategic foundations for the Kalai (1977) solution in a bargaining game with liquidity constraints. To that end, however, they use the agenda according to which bundles of goods are negotiated sequentially. In contrast, here we are interested in sequential sales of asset bundles and we obtain a new solution concept, the gradual Nash solution, in the context of decentralized asset markets.

The general equilibrium framework into which we incorporate bargaining games with an agenda corresponds to a version of the Lagos and Wright (2005) model with divisible Lucas trees, as in Geromichalos et al. (2007) and Lagos (2010).⁴ We also consider a variant where agents trade assets because of idiosyncratic valuations, as in Duffie et al. (2005). See also Lagos and Rocheteau (2009) and Üslü (2019) with unrestricted portfolios; Geromichalos and Herrenbrueck (2016), Lagos and Zhang (2020), and Wright et al. (2018), with asset trades financed with money.⁵ We are the first ones to point out the importance of the agenda of the bargaining game for qualitative and quantitative results.

Our extension with multiple assets contributes to the literature on asset price puzzles in markets with search frictions, e.g., Vayanos and Weill (2008) based on increasing-returns-to-scale matching technologies; Rocheteau (2011), Li et al. (2012) and Hu (2013) based on informational asymmetries; Lagos (2013) based on self-fulfilling beliefs in the presence of assets' extrinsic characteristics; and Geromichalos and Herrenbrueck (2016b) based on matching and bargaining friction differentials across the secondary markets where each asset

⁴In those models, the asset owner has all the bargaining power. Rocheteau and Wright (2013) adopt the proportional bargaining solution, endogenize participation, and consider non-stationary equilibria.

⁵See Trejos and Wright (2016) for a model that nests Shi (1995), Trejos and Wright (1995) and Duffie et al. (2005).

is traded. Closer to what we do, Zhu and Wallace (2007) explain the coexistence of money and interest-bearing bonds using a bargaining protocol with a two-item agenda where bargaining powers vary with the item under negotiation. In contrast to their approach, our bargaining solution has both axiomatic and strategic foundations and we do not make bargaining powers asset specific.

2 The gradual bargaining game

In this section we describe an OTC bargaining game whereby two players negotiate the sale of divisible assets in exchange for consumption goods. We set up the game and its payoffs so that it can easily be embodied into an off-the-shelf general equilibrium model of decentralized asset markets in Section 4. In this section and the next we provide a series of methodological results regarding OTC bargaining games with an agenda, their axiomatic and strategic foundations, and their positive and normative implications. This section focuses on a simple extensive-form game, called the alternating-ultimatum-offer bargaining game, and its relationship to an axiomatic solution provided by O'Neill et al. (2004). Section 3 generalizes this extensive-form game to establish a connection with the Nash bargaining solution, commonly used in the literature, and endogenizes the choice of the agenda.

The bargaining game is composed of two players, called *consumer* and *producer*, who negotiate the sale of z units of an asset in exchange for units of a commodity labeled decentralized market (DM) good.⁶ See left panel of Figure 1. The labels *consumer* and *producer* refer to agents' roles regarding the DM good. The consumer is the buyer of the DM good and the seller of the asset while the producer is the seller of the DM good and hence the buyer of the asset. The DM good is produced on the spot once an agreement is reached. We interpret $z > 0$ as the total asset holdings of the consumer that are up for sale. This quantity will be endogenized in Section 4 by allowing agents to make a portfolio choice. An outcome of the negotiation is a pair $(y, p) \in \mathbb{R}_+ \times [0, z]$ where p is the amount of assets sold for y units of the DM goods. Preferences over outcomes are represented by the following

⁶The DM good has been given different interpretations in the New Monetarist literature: a perishable consumption good or service (e.g., Lagos and Wright, 2005), physical capital (e.g., Wright et al., 2018), or an illiquid consol that is valued differently by different players (e.g., Duffie et al., 2005).

payoff functions:

$$\begin{aligned} u^b &= u(y) - p + u_0^b \\ u^s &= -v(y) + p + u_0^s, \end{aligned}$$

where u_0^b and u_0^s are the payoffs in case of disagreement (endogenized in general equilibrium later) and the superscripts b and s stand for buyer and seller of the DM good. As is standard in the search-theoretic literature on asset markets, payoffs are linear in p , hence the asset transfers utility perfectly across players up to the amount z . In contrast to p , the DM good does not transfer utility perfectly across players, i.e., in general $u'(y) \neq v'(y)$. More specifically, we assume $u'(y) > 0$, $u''(y) < 0$, $u'(0) = +\infty$, $u(0) = v(0) = v'(0) = 0$, $v'(y) > 0$, $v''(y) > 0$, and $u'(y^*) = v'(y^*)$ for some $y^* > 0$.⁷ We illustrate the determination of the players' payoffs from a trade (y^e, p^e) in the right panel of Figure 1 where disagreement points are normalized to $u_0^b = u_0^s = 0$.

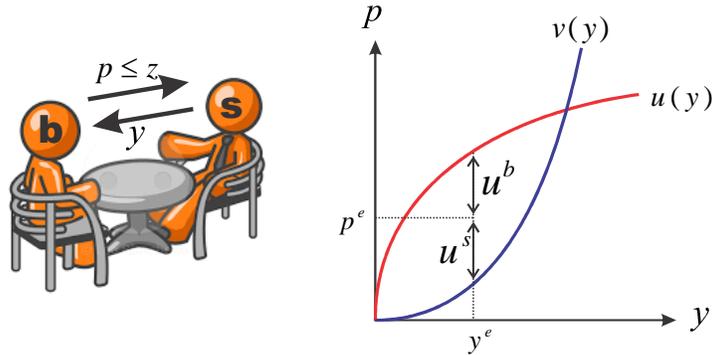


Figure 1: Left: Bilateral negotiation between consumer (b) and producer (s). Right: Payoffs of the gradual bargaining game

In the following we first propose an extensive-form game to determine (y, p) and then we adopt an axiomatic approach to show the robustness of the solution.

2.1 The alternating-ultimatum-offer bargaining game

The game has N rounds. In each round, the consumer can negotiate at most z/N units of assets for some DM output. The round-game corresponds to a two-stage ultimatum game:

⁷The Inada condition on $u(y) - v(y)$ is only needed when we incorporate the bargaining game into a general equilibrium structure. The concavity assumption makes the set of feasible utilities convex and it will allow us to obtain uniqueness of the general equilibrium later.

in the first stage an offer is made; in the second stage the offer is accepted or rejected.⁸ In order to maintain some symmetry between the two players (when N is large), the identity of the proposer alternates across rounds.⁹ We assume N is even and the producer is the one making the first offer. These assumptions will be inconsequential when we consider the limit as N becomes large. The game tree is represented in Figure 2.

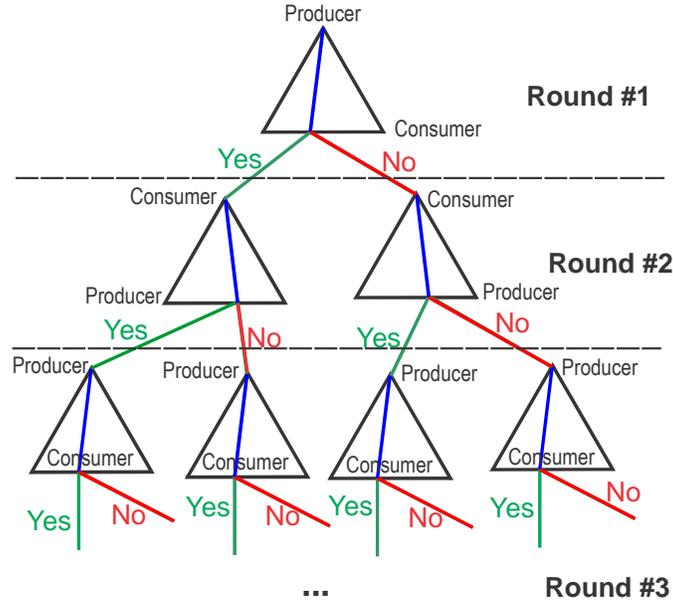


Figure 2: Game tree of the alternating-ultimatum-offer game

In order to solve for the equilibrium, it is useful to introduce an explicit notion of time in the negotiation, denoted by τ . We map asset holdings into time by assuming that $\delta > 0$ units of asset can be negotiated per unit of time. Hence, $\tau \equiv nz/(\delta N)$ is the time at the end of the n^{th} round of the negotiation (in each of the n rounds, z/N assets are up for negotiation, and each asset takes $1/\delta$ units of time to be negotiated). We will rely heavily on δ in our general equilibrium model with multiple assets of Section 5. The utility accumulated by the consumer up to time τ is

$$u^b(\tau) = u[y(\tau)] - p(\tau) + u_0^b, \quad (1)$$

⁸A feature of our game is that if an offer is rejected, the z/N units of assets that are unsold cannot be renegotiated later in the game. While this assumption is no different from the one in standard ultimatum games (i.e., agents are committed to the rules of the game), the solution to our game, however, is robust to this feature, i.e., the game could include more than N rounds to allow for some amount of renegotiation. See Appendix B.

⁹Our game resembles the finite bargaining game with alternating offers of Stahl (1972). It differs from it in that players are negotiating different items in each round.

where $y(\tau)$ is the consumer's cumulative consumption at time τ , $p(\tau)$ is his cumulative payment with the asset. The utility accumulated by the producer up to τ is

$$u^s(\tau) = -v[y(\tau)] + p(\tau) + u_0^s. \quad (2)$$

Given the feasibility constraint $p(\tau) \leq \delta\tau$, we can define a Pareto frontier for each τ , i.e.,

$$u^b = \max_{y, p \leq \delta\tau} \{u(y) - p + u_0^b\} \quad \text{s.t.} \quad -v(y) + p + u_0^s \geq u^s.$$

These Pareto frontiers play a key role to solve for the SPE of the game by backward induction.

Lemma 1 (Pareto frontiers) *The Pareto frontier at time τ satisfies $H(u^b, u^s, \tau) = 0$ where*

$$H(u^b, u^s, \tau) = \begin{cases} u(y^*) - v(y^*) - (u^b - u_0^b) - (u^s - u_0^s) & \text{if } u^s - u_0^s \leq \delta\tau - v(y^*) \\ \delta\tau - v[u^{-1}(\delta\tau + u^b - u_0^b)] - (u^s - u_0^s) & \text{otherwise.} \end{cases} \quad (3)$$

The function H is continuously differentiable, increasing in τ (strictly so if $y < y^*$), decreasing in u^b and u^s . Consequently, each Pareto frontier has a negative slope:

$$\left. \frac{\partial u^s}{\partial u^b} \right|_{H(u^b, u^s, \tau)=0} = \begin{cases} -1 & \text{if } u^s - u_0^s \leq \delta\tau - v(y^*) \\ -\frac{v'(y)}{u'(y)} & \text{otherwise.} \end{cases}$$

The Pareto frontier is linear when $y = y^*$. When $y < y^*$, it is strictly concave.

We call a bargaining round an *active round* if there is trade. We say that a SPE is *simple* if in each active round the consumer offers z/N units of assets, except possibly for the last active round, and active rounds are followed by inactive rounds (if any).

Proposition 1 (SPE of the alternating-ultimatum-offer game.) *All SPE of the alternating-ultimatum-offer game share the same final payoffs, $(\tilde{u}_N^b, \tilde{u}_N^s)$, corresponding to the last term of the sequence, $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=0}^N$, with $(\tilde{u}_0^b, \tilde{u}_0^s) = (u_0^b, u_0^s)$, and*

$$H(\tilde{u}_j^b, \tilde{u}_{j-1}^s, jz/N) = 0 \quad \text{and} \quad \tilde{u}_j^s = \tilde{u}_{j-1}^s, \quad \text{for } j \geq 1 \text{ odd,} \quad (4)$$

$$H(\tilde{u}_{j-1}^b, \tilde{u}_j^s, jz/N) = 0 \quad \text{and} \quad \tilde{u}_j^b = \tilde{u}_{j-1}^b, \quad \text{for } j \geq 2 \text{ even.} \quad (5)$$

If the final y is less than y^ , then the SPE is unique and simple; otherwise, there is a unique simple SPE. Moreover, in any simple SPE, the intermediate payoffs, $\{(u_n^b, u_n^s)\}_{n=1,2,\dots,N}$, con-*

verge to the solution, $\langle u^b(\tau), u^s(\tau) \rangle$, to the following differential equations as N approaches $+\infty$:

$$u^{b'}(\tau) = -\frac{1}{2} \frac{\partial H(u^b, u^s, \tau) / \partial \tau}{\partial H(u^b, u^s, \tau) / \partial u^b} \quad (6)$$

$$u^{s'}(\tau) = -\frac{1}{2} \frac{\partial H(u^b, u^s, \tau) / \partial \tau}{\partial H(u^b, u^s, \tau) / \partial u^s}. \quad (7)$$

Proposition 1 (proved in Appendix B) establishes that the SPE of the alternating-ultimatum-offer game is essentially unique — any multiplicity when $y = y^*$ is due to differences in the timing of asset sales that are payoff-irrelevant. According to (4)-(5) the terminal payoffs are obtained as the final terms of a simple recursion whereby in any odd periods j the producer's utility is unchanged relative to the previous round, $j - 1$, and the consumer's payoff is chosen so that the pair of utilities belong to the Pareto frontier corresponding to the asset holdings jz/N . Thus, the consumer gets the full surplus in odd rounds while the producer receives full surplus in even rounds.

When N approaches $+\infty$, i.e., when bargaining becomes gradual, equilibrium payoffs are characterized by the system of differential equations, (6)-(7). The interpretation of this solution is as follows. An increase in τ by one unit expands the bargaining set by $\partial H / \partial \tau$. The maximum utility gain that the consumer could enjoy from this expansion is $-(\partial H / \partial \tau) / (\partial H / \partial u^b)$, as illustrated by the horizontal arrow in Figure 3. According to (6), the consumer enjoys half of this gain. The same holds true for the producer. By combining (6) and (7), the slope of the gradual agreement path is:

$$\frac{\partial u^s}{\partial u^b} = \frac{\partial H(u^b, u^s, \tau) / \partial u^b}{\partial H(u^b, u^s, \tau) / \partial u^s}. \quad (8)$$

According to (8), the slope of the gradual bargaining path is equal to the opposite of the slope of the Pareto frontier.

The proof of Proposition 1 consists of two steps: first, we characterize the SPE for any (sub)game with an arbitrary number of remaining rounds, J . In the second part, we establish that the sequence of intermediate payoffs of the SPE converges to the solution to the system of differential equations, (6) and (7), as N approaches $+\infty$. The intuition goes as follows. Suppose the negotiation enters its last round, N , and the two agents have agreed upon some intermediate payoffs (u_{N-1}^b, u_{N-1}^s) . The consumer makes the last take-

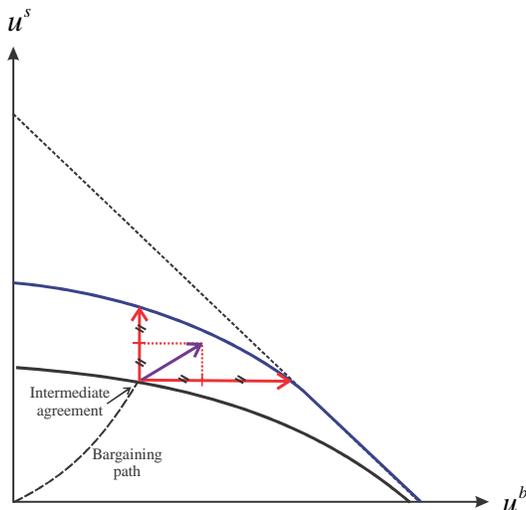


Figure 3: Solution to a gradual bargaining problem

it-or-leave offer, which maximizes his payoff by keeping the producer's payoff unchanged at u_{N-1}^s . Graphically, the final payoffs are constructed from the intermediate payoffs by moving horizontally from the lower Pareto frontier, to which (u_{N-1}^b, u_{N-1}^s) belongs, to the upper Pareto frontier corresponding to an increase in assets of z/N , as shown in the left panel of Figure 4.

We now move backward in the game by one round. Suppose that the negotiation enters round $N - 1$ with some intermediate payoffs, (u_{N-2}^b, u_{N-2}^s) , with the producer making the offer. Now, if the consumer rejects the producer's offer, the negotiation enters its last round and the consumer's payoff is obtained as before, i.e., by moving horizontally from the lower frontier to the upper frontier. Given the consumer's payoff, the producer's payoff is obtained such that the pair of payoffs is located on the last Pareto frontier. Graphically, there is first a horizontal move from the initial payoff, (u_{N-2}^b, u_{N-2}^s) , to the next Pareto frontier that determines the consumer's terminal payoff, and then a vertical move to the following frontier that determines the producer's payoff, u_N^s , as shown in the right panel of Figure 4. We iterate this procedure backward until we reach the start of the game with initial payoffs (u_0^b, u_0^s) .

Once we have the terminal payoffs, we use another backward induction to determine the sequence of intermediate payoffs. The intermediate payoffs at the end of the $(N - 1)^{\text{th}}$ round lie on the $(N - 1)^{\text{th}}$ frontier and are obtained by moving horizontally from the N^{th} frontier to the $(N - 1)^{\text{th}}$ frontier since the consumer is making the last offer. The intermediate payoffs on the $(N - 2)^{\text{th}}$ frontier are obtained by moving first vertically, from the N^{th} frontier to the

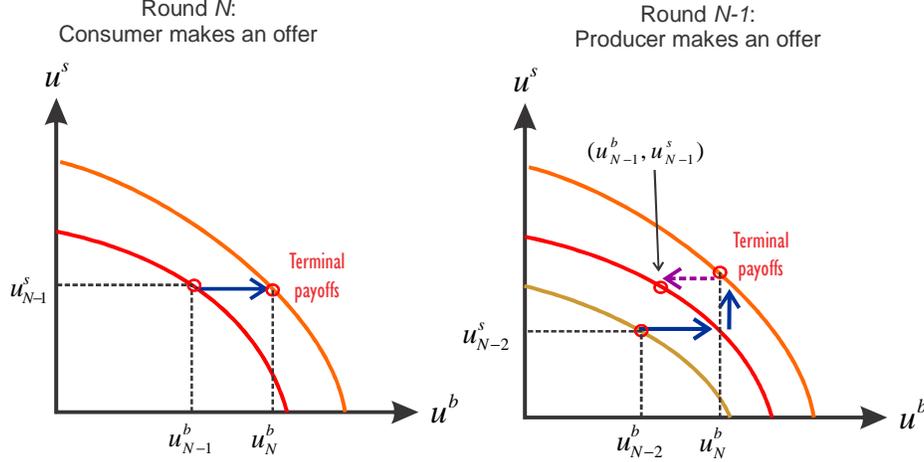


Figure 4: Left panel: offer in last round; Right panel: offer in $(N - 1)^{\text{th}}$ round

$(N - 1)^{\text{th}}$ frontier, and then horizontally from the $(N - 1)^{\text{th}}$ frontier to the $(N - 2)^{\text{th}}$ frontier by using the same reasoning as above. It turns out that the two sequences constructed above get closer to one another as N becomes large, and, both converge to the gradual bargaining path according to (8).

We now characterize in closed-form the final allocations. From (4)-(5) the final outcome corresponds to the last term of the sequence $\{(y_n, p_n)\}_{n=0}^N$ computed recursively from $(y_0, p_0) = (0, 0)$ as follows:

$$(y_n, p_n) \in \arg \max \{u(y_n) - u(y_{n-1}) - (p_n - p_{n-1})\} \quad \text{if } n \text{ odd} \quad (9)$$

$$(y_n, p_n) \in \arg \max \{(p_n - p_{n-1}) - [v(y_n) - v(y_{n-1})]\} \quad \text{if } n \text{ even}, \quad (10)$$

where each maximization problem is subject to the participation constraints,

$$v(y_n) - v(y_{n-1}) \leq p_n - p_{n-1} \leq u(y_n) - u(y_{n-1}), \quad (11)$$

and the feasibility condition,

$$p_n - p_{n-1} \leq \frac{z}{N}. \quad (12)$$

In odd periods (y_n, p_n) corresponds to a take-it-or-leave-it offer by the consumer whereas in even periods it coincides with a take-it-or-leave-it offer by the producer. If (12) binds in each

round, then the solution is:

$$y_n = v^{-1} \left[\frac{z}{N} + v(y_{n-1}) \right] \quad \text{if } n \text{ odd} \quad (13)$$

$$y_n = u^{-1} \left[\frac{z}{N} + u(y_{n-1}) \right] \quad \text{if } n \text{ even.} \quad (14)$$

So, the solution y_N can easily be computed given the initial condition, $y_0 = 0$.

2.2 Negotiated price and trade size

We now turn to the implications of the gradual bargaining solution for asset prices and trade sizes and focus on the limit case where N approaches infinity. We derive in closed-form a payment function, $p(y)$, that specifies the quantity of assets required to purchase y units of goods, and that plays a critical role in models of asset liquidity (e.g., Lagos et al., 2017). From the definition of H in (3), the solution to the bargaining game, (6)-(7), can be reexpressed as

$$u^{b'}(\tau) = \delta \frac{u'(y) - v'(y)}{2v'(y)} \quad (15)$$

$$u^{s'}(\tau) = \delta \frac{u'(y) - v'(y)}{2u'(y)}, \quad (16)$$

if $\delta\tau < u^s - u_0^s + v(y^*)$, and $u^{b'}(\tau) = u^{s'}(\tau) = 0$ otherwise. From (15) and (16) the slope of the gradual bargaining path is $\partial u^s / \partial u^b = v'(y)/u'(y)$, which is increasing in y , i.e., it becomes steeper as the negotiation progresses. The producer's share in the match surplus increases throughout the negotiation as the gap between $u'(y)$ and $v'(y)$ shrinks over time.

Proposition 2 (*Prices and trade sizes*) *Along the gradual bargaining path, the price of the asset in terms of DM goods is*

$$\frac{y'(\tau)}{\delta} = \frac{1}{2} \left(\overbrace{\frac{1}{v'(y)}}^{\text{bid price}} + \overbrace{\frac{1}{u'(y)}}^{\text{ask price}} \right) \quad \text{for all } y < y^*. \quad (17)$$

The overall payment for y units of consumption is

$$p(y) = \int_0^y \frac{2v'(x)u'(x)}{u'(x) + v'(x)} dx. \quad (18)$$

If $z \geq p(y^*)$ then $y = y^*$ and $y = p^{-1}(z)$ otherwise.

According to (17), the negotiated price is the arithmetic average of the bid and ask prices. The bid price of one unit of asset at time τ , i.e., the maximum price in terms of DM goods that the producer is willing to pay to acquire it, is equal to $1/v'(y)$. The ask price at time τ , i.e., the minimum price in terms of DM goods that the consumer is willing to accept to give up the asset, is $1/u'(y)$. The bid price decreases with y because the producer incurs a convex cost to finance an additional unit of asset. The ask price increases with y because the consumer enjoys a decreasing marginal utility in exchange of an additional unit of asset. So the negotiated price can be non-monotone with the size of the trade.

The payment function, (18), can be rewritten as

$$p(y) = \int_0^y \frac{v'(x)}{u'(x) + v'(x)} u'(x) dx + \int_0^y \frac{u'(x)}{u'(x) + v'(x)} v'(x) dx.$$

It is reminiscent of the payment function obtained from the Nash solution (e.g., Lagos and Wright, 2005) where $p^{\text{Nash}}(y) = [1 - \Theta(y)] u(y) + \Theta(y)v(y)$ and $\Theta(x) \equiv u'(x)/[u'(x) + v'(x)]$ is interpreted as the consumer's share in the surplus of the match. In order to make the connection clearer, we integrate $p(y)$ by parts to obtain:

$$p(y) = \overbrace{[1 - \Theta(y)] u(y) + \Theta(y)v(y)}^{\text{Nash}} + \int_0^y \Theta'(x) [u(x) - v(x)] dx.$$

So the payment function under gradual bargaining is the sum of the payment function under Nash bargaining and an additional term that is negative since $\Theta'(x) < 0$. This additional term takes into account the change in the consumer's share over the gradual negotiation, i.e., as the negotiation advances the consumer's share decreases. We will come back to this comparison in the next section. It is worth noticing that $p(y)$ is independent of δ , and hence the outcome of the negotiation does not depend on the time it takes to negotiate assets sequentially: only N matters for the outcome. We will make δ relevant in Section 5 by assuming that the negotiation has a stochastic time horizon.

From (18) we can compute the consumer's surplus from a trade:

$$u(y) - p(y) = \int_0^y \frac{u'(x) [u'(x) - v'(x)]}{u'(x) + v'(x)} dx, \quad \text{for all } y \leq y^*.$$

The surplus increases with y , is strictly concave for all $y < y^*$, and is maximized at $y = y^*$. We will emphasize the importance of the monotonicity of the surplus for individual choices of asset holdings and asset prices later when we turn to the general equilibrium.

2.3 Asymmetric agenda

So far the agenda of the negotiation corresponds to a uniform partition of the portfolio, $[0, z]$, where each asset bundle has the same size, z/N . In the following we modify the agenda to provide a non-cooperative foundation for asymmetric bargaining powers. Such asymmetric solutions are useful in many applications to decentralized asset markets with endogenous participation and investment decisions. We still assume that N is even. In each round where the consumer is making the offer, the amount of assets that can be negotiated is $2\theta z/N$ where $\theta \in [0, 1]$. In rounds where the producer is making the offer, the amount of assets up for negotiation is $2(1 - \theta)z/N$. Note that $\theta = 1/2$ corresponds to the bargaining game studied earlier. We show in Appendix B that the solution to this bargaining game generalizes (6)-(7) as follows:

$$u^b(\tau) = -\theta \frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^b} \quad (19)$$

$$u^s(\tau) = -(1 - \theta) \frac{\partial H(u^b, u^s, \tau)/\partial \tau}{\partial H(u^b, u^s, \tau)/\partial u^s}, \quad (20)$$

where $\theta \in [0, 1]$ is interpreted as the consumer's bargaining power.¹⁰ By the same reasoning as above, the DM price of assets evolves according to

$$\frac{y'(\tau)}{\delta} = \left(\theta \overbrace{\frac{1}{v'(y)}}^{\text{bid price}} + (1 - \theta) \overbrace{\frac{1}{u'(y)}}^{\text{ask price}} \right). \quad (21)$$

It is now a weighted average of the bid and ask prices where the weights are given by the relative bargaining powers of the consumer and the producer. From (21) the DM price of the asset is increasing in θ . The payment for y units of DM consumption is

$$p(y) = \int_0^y \frac{u'(x) v'(x)}{\theta u'(x) + (1 - \theta) v'(x)} dx \quad \text{for all } y \leq y^*. \quad (22)$$

¹⁰This solution coincides with the axiomatic solution of Wiener and Winter (1998). One could make the bargaining power a function of time, τ , or output traded, y , without affecting the results significantly.

2.4 An axiomatic approach

An axiomatic approach, by abstracting from the details of the bargaining game, provides a sense of the robustness of our solution.¹¹ The Nash definition of a bargaining problem, which does not include the notion of agenda, was extended by O’Neill et al. (2004). The agenda takes the form of a family of feasible sets indexed by time. The difficulty for applications to decentralized asset markets is to identify the relevant agenda for the problem at hand. In the context of our model where agents negotiate gradually the sale of assets, the bargaining problem is defined as follows.

Definition 1 *A gradual bargaining problem between a consumer holding z units of asset and a producer is a collection of Pareto frontiers, $\langle H(u^b, u^s, \tau) = 0, \tau \in [0, z/\delta] \rangle$ and a pair of disagreement points, (u_0^b, u_0^s) .*

A gradual agreement path is a function, $o : [0, z/\delta] \rightarrow \mathbb{R}_+ \times [0, z]$, that specifies an allocation (y, p) for all $\tau \in [0, z/\delta]$ and associated utility levels, $\langle u^b(\tau), u^s(\tau) \rangle$. The gradual solution of O’Neill et al. (2004) is the unique solution to satisfy five axioms: Pareto optimality, scale invariance, symmetry, directional continuity, and time-consistency. The first three axioms are axioms imposed by Nash (1950). The last two axioms are specific to the new definition of the bargaining problem. Directional continuity imposes a notion of continuity for the bargaining path with respect to changes in the agenda. The requirement of time-consistency specifies that if the negotiation were to start with the agreement reached at time τ as the new disagreement point, then the bargaining path onward would be unchanged. Theorem 1 of O’Neill et al. (2004) applied to the bargaining problem above leads to the following result.

Theorem 1 (Ordinal solution of O’Neill et al., 2004) *There is a unique solution to the gradual bargaining problem defined by $\langle H(u^b, u^s, \tau) = 0, \tau \in [0, z/\delta], (u_0^b, u_0^s) \rangle$ that satisfies*

¹¹As written by Serrano (2008) in his description of the Nash program:

The non-cooperative approach to game theory provides a rich language and develops useful tools to analyze strategic situations. One clear advantage of the approach is that it is able to model how specific details of the interaction may impact the final outcome. One limitation, however, is that its predictions may be highly sensitive to those details. For this reason it is worth also analyzing more abstract approaches that attempt to obtain conclusions that are independent of such details. The cooperative approach is one such attempt.

the five axioms of Pareto optimality, scale invariance, symmetry, directional continuity, and time-consistency. This solution coincides with (6)-(7).

The equilibrium payoffs of the alternating-ultimatum-offer bargaining game coincide with the axiomatic solution from O'Neill et al. (2004). While scale invariance was imposed as an axiom, the solution exhibits ordinality endogenously: the solution is covariant with respect to any order-preserving transformation. This result is noteworthy because Shapley (1969) shows that for standard Nash problems with two players, no single-valued solution can satisfy Pareto efficiency, symmetry, and ordinality. Finally, if the axiom of symmetry is dropped, then the generalized ordinal solutions corresponding to the bargaining problem in Definition 1 solve (19)-(20).

3 Relation to Nash bargaining

The game studied in Section 2.1 is extended so that each round, $n \in \{1, \dots, N\}$, is composed of an unbounded number of stages during which the two players bargain over z/N units of assets following an alternating-offer protocol as in Rubinstein (1982). The consumer is the first proposer if n is odd, and the producer is the first proposer otherwise. The round-game, illustrated in Figure 5, is as follows. In the initial stage, the first proposer makes an offer and the other agent either accepts it or rejects it. If the offer is accepted, round n ends and agents move to round $n+1$. If the offer is rejected then there are two cases. With probability $(1 - \xi_n)$ round n is terminated and the players move to round $n+1$ without having reached an agreement. With probability ξ_n the negotiation continues and the responder becomes the proposer in the following stage. We focus on the limit case where ξ_n converges to one, and the order of convergence is from ξ_N to ξ_1 .

Proposition 3 (Repeated Rubinstein game.) *There exists a SPE of the repeated Rubinstein game when taking limits according to the order $\xi_N \rightarrow 1, \xi_{N-1} \rightarrow 1, \dots, \xi_1 \rightarrow 1$, characterized by a sequence of intermediate allocations, $\{(y_n, p_n)\}_{n=0}^N$, solution to:*

$$(y_n, p_n) \in \arg \max_{y, p} [u(y) - p - u(y_{n-1}) + p_{n-1}] [-v(y) + p + v(y_{n-1}) - p_{n-1}]$$

$$s.t. \quad p \leq \frac{nz}{N}, \quad (23)$$

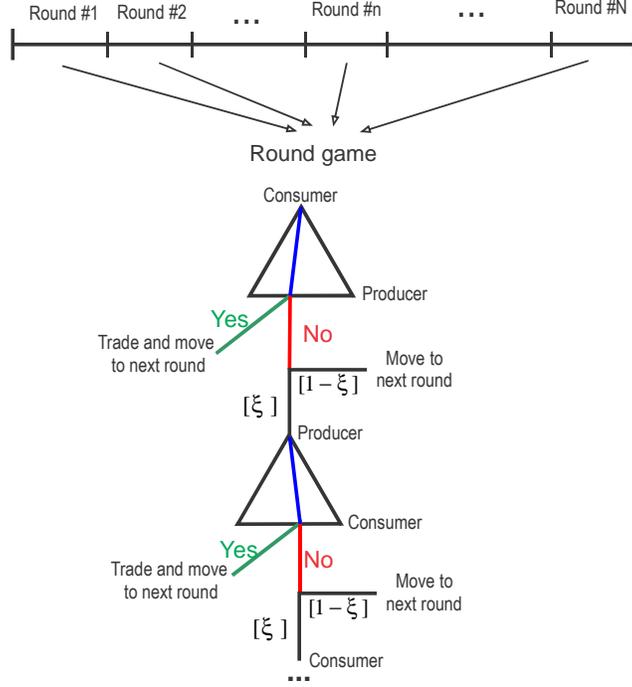


Figure 5: Game tree with alternating offers in each round

for all $n \in \{1, \dots, N\}$ with $(y_0, p_0) = (0, 0)$. As $N \rightarrow +\infty$ the solution converges to the solution of the alternating-ultimatum-offer game characterized in Proposition 2.

The intermediate allocation at the end of each round, given by (23), maximizes the Nash product of agents' surpluses where the endogenous disagreement points are the intermediate payoffs of the previous round. The proof (in Appendix C) is based on backward induction. Consider the last round with some intermediate agreement (u_{N-1}^b, u_{N-1}^s) . The outcome of the Rubinstein game as the risk of breakdown goes to zero corresponds to the Nash solution with disagreement point (u_{N-1}^b, u_{N-1}^s) . Next, consider round $N-1$ with intermediate payoffs (u_{N-2}^b, u_{N-2}^s) . The relevant disagreement points, $(\tilde{u}_{N-1}^b, \tilde{u}_{N-1}^s)$, are given by the outcome of the negotiation in round N if there is no agreement in round $N-1$, i.e., $(\tilde{u}_{N-1}^b, \tilde{u}_{N-1}^s)$ maximizes the Nash product $(\tilde{u}_{N-1}^b - u_{N-2}^b)(\tilde{u}_{N-1}^s - u_{N-2}^s)$. Given $(\tilde{u}_{N-1}^b, \tilde{u}_{N-1}^s)$, the negotiation in round $N-1$, in which players are forward looking, determines the final payoffs. As the risk of breakdown vanishes, these payoffs, (u_N^b, u_N^s) , coincide with the Nash solution, i.e., they maximize $(u_N^b - \tilde{u}_{N-1}^b)(u_N^s - \tilde{u}_{N-1}^s)$. For any given initial condition (u_0^b, u_0^s) , this iterative procedure pins down the terminal payoffs. Once terminal payoffs are determined, we use a second backward induction to find the sequence of intermediate payoffs. Intermediate

payoffs in round $N - 1$, (u_{N-1}^b, u_{N-1}^s) , correspond to the disagreement points of the Nash solution that generates the terminal payoffs, i.e., $(u_{N-1}^b, u_{N-1}^s) = (\tilde{u}_{N-1}^b, \tilde{u}_{N-1}^s)$. And so on. The determination of payoffs is illustrated in Figure 6.

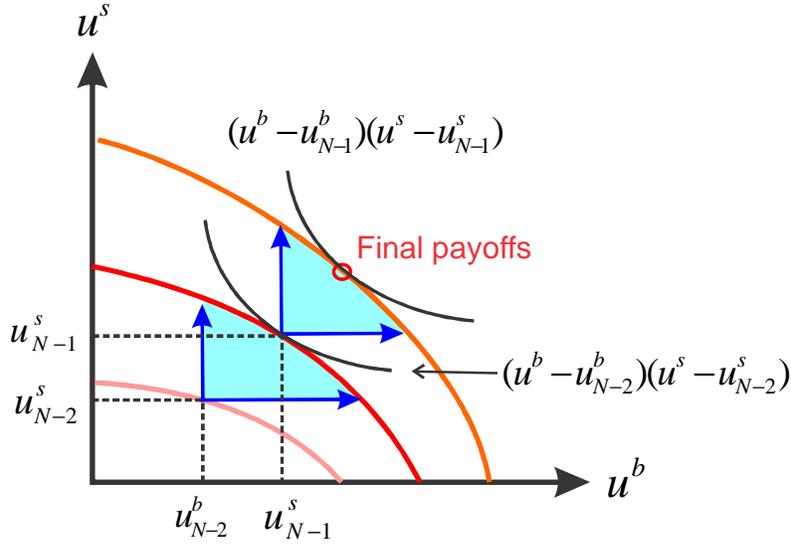


Figure 6: Computing terminal payoffs from round $N - 1$

In order to fix the intuition, suppose $N = 2$. In the first round of the negotiation agents negotiate (y_1, p_1) taking into account that the outcome of the second round, (y_2, p_2) , is a function of the interim agreement, (y_1, p_1) . In case of disagreement, the players negotiate in a single round the remaining $z/2$ assets so that the allocation is given by

$$(\tilde{y}_1, \tilde{p}_1) \in \arg \max_{y,p} [u(y) - p] [-v(y) + p] \quad \text{s.t.} \quad p \leq \frac{z}{2}. \quad (24)$$

Because agents in the first round can anticipate the outcome of the second round, they are negotiating the final outcome, (y, p) where $y = y_1 + y_2(y_1, p_1)$ and $p = p_1 + p_2(y_1, p_1)$. Hence, the final outcome is given by

$$(y, p) \in \arg \max_{y,p} [u(y) - p - u(\tilde{y}_1) + \tilde{p}_1] [-v(y) + p + v(\tilde{y}_1) - \tilde{p}_1] \quad \text{s.t.} \quad p \leq z. \quad (25)$$

In the second round, agents solve the same problem where the disagreement point correspond to the actual trade in the first round, (y_1, p_1) , i.e.,

$$(y, p) \in \arg \max_{y,p} [u(y) - p - u(y_1) + p_1] [-v(y) + p + v(y_1) - p_1] \quad \text{s.t.} \quad p - p_1 \leq \frac{z}{2}. \quad (26)$$

The first round outcome is chosen so that (25) and (26) hold and it turns out that $(y_1, p_1) = (\tilde{y}_1, \tilde{p}_1)$. The interim trade in the first round corresponds to the trade that would take place in case of disagreement. The total surplus in both rounds is equal to $[u(y) - v(y)] - [u(\tilde{y}_1) - v(\tilde{y}_1)]$, i.e., agents are negotiating the marginal surplus in each round. It is easy to generalize the logic to $N > 2$. For instance, if $N = 3$ then the final outcome is negotiated in round 1 by forward-looking agents according to the Nash solution with disagreement points corresponding to the solution of (25).

From (23) the intermediate allocations, $\{(y_n, p_n)\}_{n=0}^N$, solve:

$$\int_{y_{n-1}}^{y_n} \frac{v'(y_n)u'(x) + u'(y_n)v'(x)}{u'(y_n) + v'(y_n)} dx \leq \frac{z}{N} \quad " = " \text{ if } y_n < y^*, \quad (27)$$

$$p_n - p_{n-1} = \min \left\{ \frac{[u(y^*) - u(y_{n-1})] + [v(y^*) - v(y_{n-1})]}{2}, \frac{z}{N} \right\},$$

with $y_0 = 0$. From (27), when the liquidity constraint, $p_n \leq nz/N$, binds, then the payment for $y_n - y_{n-1}$ units of DM goods is equal to a weighted sum of the marginal utilities of consumption and the marginal disutilities of production. If $N = 1$ then (27) corresponds to symmetric Nash.¹² Summing (27) across n and taking the limit as N goes to $+\infty$ gives the gradual solution.

In the following proposition we let consumers (asset owners) choose the number of rounds of the negotiation, N . The key observation from (27) is that the consumer's share in the surplus of the n^{th} round, $u'(y_n)/[u'(y_n) + v'(y_n)]$, decreases with y_n .

Proposition 4 (*Optimal gradualism*) *Consumers obtain their highest surplus by negotiating the sale of their assets one infinitesimal unit at a time, $N = +\infty$.*

The agenda underlying the Nash solution ($N = 1$) is suboptimal from the standpoint of asset owners. They strictly prefer to sell their assets gradually over time. The consumer's gain from bargaining gradually is

$$p_1(y) - p_\infty(y) = \int_0^y \left[\frac{v'(y)}{u'(y) + v'(y)} - \frac{v'(x)}{u'(x) + v'(x)} \right] [u'(x) - v'(x)] dx,$$

where $p_1(y)$ is the amount of assets in exchange for y units of DM goods if the negotiation takes place in a single round, which implements the Nash solution. Under Nash bargain-

¹²In the Appendix D, we study a version of the game that implements the generalized Nash solution.

ing the producer’s share in each increment of the match surplus is constant and equal to $v'(y)/[u'(y) + v'(y)]$, which is larger than the variable share, $v'(x)/[u'(x) + v'(x)]$ for all $x < y$, under gradual bargaining. Intuitively, selling all the assets at once has a negative impact on the consumer’s surplus share that can be reduced by selling them through small quantities — a form of dynamic price discrimination.

In order to make this intuition clearer, let’s compare the consumer’s surplus when $N = 2$ and when $N = 1$. From (25) if the agents decide to go from $N = 1$ to $N = 2$, the only change in the Nash product are the disagreement points that increase from $(0, 0)$ to $[u(\tilde{y}_1) - \tilde{p}_1, v(\tilde{y}_1) - \tilde{p}_1]$. Let $\Theta(y_{N=1}) \equiv u'(y_{N=1})/[u'(y_{N=1}) + v'(y_{N=1})]$ be the consumer’s share in the total match surplus if the negotiation takes place in one round. If $u(\hat{y}_1) - \hat{p}_1 = \Theta(y_{N=1}) [u(\hat{y}_1) - v(\hat{y}_1)]$, then the change in disagreement points has no effect on the final agreement. However, $\Theta(\hat{y}_1) > \Theta(y_{N=1})$ due to the strict concavity of $u(y) - v(y)$. The consumer can extract a larger surplus by negotiating in two steps.

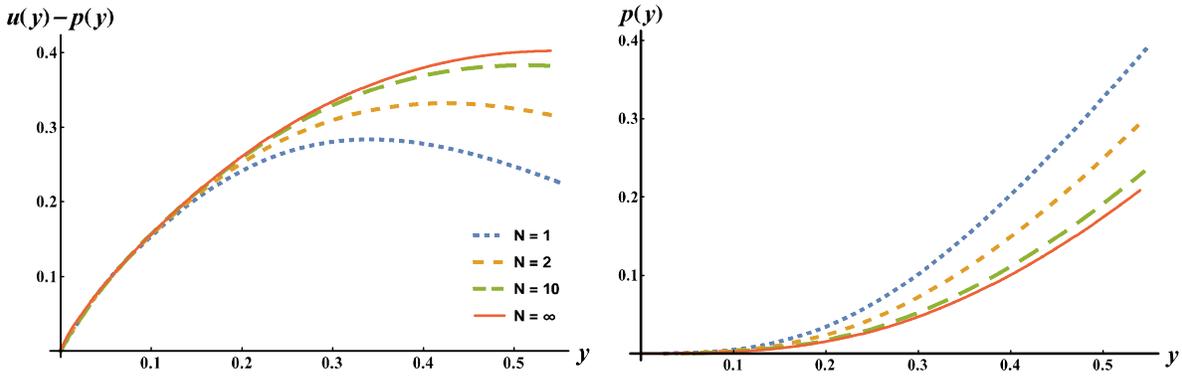


Figure 7: Consumer surplus and payment as a function of trade size for different N .

Figure 7 plots the final payment and the consumer’s surplus as a function of trade size for games with $N \in \{1, 2, 10, +\infty\}$. The larger the number of rounds, the lower the payment, and the higher the consumer’s surplus for any trade size, y . For finite values of N the consumer’s surplus is non-monotone in y . Moreover, the value of y that maximizes the consumer’s surplus increases with N . The monotonicity (or lack thereof) of the consumer’s surplus will have important normative implications when we study the general equilibrium of the economy. To explore these implications, in what follows whenever we refer to N -round games we are using this repeated Rubinstein game.

4 Bargaining and markets

Sections 2 and 3 provided the methodological tools to analyze OTC bargaining games with an agenda. The games we studied took as given the asset holdings that were up for negotiation (z) and omitted intertemporal considerations, such as the opportunity cost of holding assets across periods, that are critical for portfolio choices and allocations in decentralized markets. We now move to the general equilibrium analysis of decentralized asset markets and provide a user-friendly guide of bargaining solutions with an agenda in this context.

In terms of economic insights, we study the implications of gradualism to determine terms of trade for asset prices, allocations, and welfare. While Proposition 4 established that it is optimal for asset owners to sell their assets gradually, we will now demonstrate that gradual negotiations lead to allocations that are superior from a social welfare perspective. We will provide a stark example of an asset market based on a simplified version of Lagos and Zhang (2020) where Nash bargaining generates the worst possible allocation whereas gradual bargaining generates the first best.

4.1 General equilibrium setting

The environment is based on the workhorse model of monetary theory of Lagos and Wright (2005).¹³ The population of agents is divided evenly between a unit measure of consumers and a unit measure of producers. There is an infinite (countable) number of periods, where each period is divided into two stages. The first stage is the decentralized market studied earlier where agents trade goods and assets in pairwise meetings formed at random. The measure of bilateral matches is $\alpha \in (0, 1]$. The second stage, labeled CM (for centralized market), features a centralized Walrasian market. It is in this second stage that agents choose their asset holdings, z , by taking prices and rates of return parametrically. There is one good in each stage and we take the CM good as numéraire. The timing within a representative period is illustrated in Figure 8.

Consumers' preferences are represented by the period utility function, $u(y) - h$, where y is the DM good traded in pairwise meetings in stage 1 and h is the disutility of producing h

¹³We adopt the version with two distinct types of agents as in Lagos and Rocheteau (2005) and Rocheteau and Wright (2005). For various treatments of the New Monetarist model, see Rocheteau and Nosal (2017) and Lagos et al. (2017).

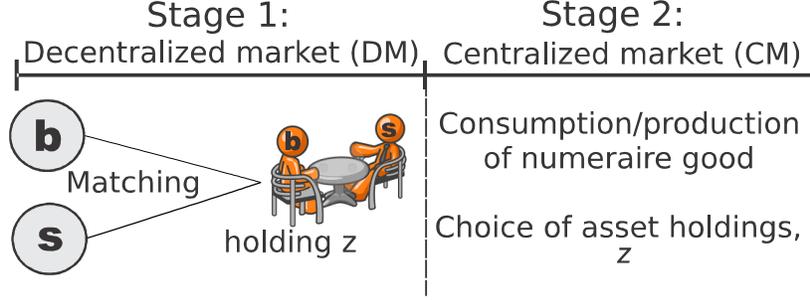


Figure 8: Timing of a representative period

units of numéraire in stage 2. Producers' preferences are represented by $-v(y) + c$, where c is the consumption of the numéraire in stage 2. Recall that y^* is the quantity that maximizes gains from trade in pairwise meetings, $u'(y^*) = v'(y^*)$. Note also that all agents' utilities are linear in the numéraire good, which is consistent with the quasi-linear payoffs of the bargaining game in Section 2. All agents share the same discount factor across periods, $\beta \equiv (1 + \rho)^{-1} \in (0, 1)$.

Agents, who are anonymous, cannot issue private IOUs. This assumption creates a need for liquid assets. As in Lagos (2010) and Geromichalos et al. (2007) there is an exogenous measure A of long-lived Lucas trees that are perfectly durable, storable at no cost, and non-counterfeitable. All trees are identical and one unit of tree pays off $d \geq 0$ units of numéraire at the start of the CM. Fiat money is a special case where $d = 0$. For that special case we allow the supply of the asset to grow at a constant rate, π , through lump-sum transfers or taxes to either buyers or sellers. We denote ϕ_t the competitive (ex dividend) price of Lucas trees in the CM in terms of the numéraire. Hence, if an agent holds a units of Lucas trees at the beginning of a period, his asset holdings expressed in terms of the numéraire are $z = a(\phi_t + d)$. In pairwise meetings, agents bargain gradually according to the strategic game or axiomatic solution described in Section 2.3 where the consumer's bargaining power is $\theta \in [0, 1]$.

In order to fix ideas, a preview of trade patterns in equilibrium is as follows. Consumers in pairwise meetings consume some endogenous quantity y in exchange for some endogenous quantity of assets. Producers in pairwise meetings produce y in exchange for assets. In the second stage, roles are reversed: consumers replenish their asset holdings by producing the numéraire good with their own labor while producers sell the assets received in the first stage in exchange for the numéraire good to consume.

4.2 Asset prices and welfare

We restrict our attention to stationary equilibria where the price of Lucas trees is constant at ϕ and hence their gross rate of return is also constant and equal to $R = 1 + r = (\phi + d)/\phi$. In the case of fiat money, $R = \phi_{t+1}/\phi_t$, is equal to the inverse of the gross growth rate of the money supply, $1/(1 + \pi)$.

Value functions The lifetime expected utility of a consumer (i.e., buyer of DM goods) with wealth z in the CM is

$$W^b(z) = \max_{z', h} \{-h + \beta V^b(z')\} \quad \text{s.t.} \quad z' = R(z + h), \quad (28)$$

where z' are next-period asset holdings, and $V^b(z')$ is the value function at the start of the DM. From (28) the consumer chooses his production of numéraire and future asset holdings in order to maximize his discounted continuation value net of the disutility of production. According to the budget constraint, next-period asset holdings are equal to current asset holdings plus output from production, everything multiplied by the gross rate of return of assets. Substituting h by its expression coming from the budget identity into the objective, we obtain

$$W^b(z) = z + \max_{z' \geq 0} \left\{ -\frac{z'}{R} + \beta V^b(z') \right\}. \quad (29)$$

As is standard, W^b is linear in wealth. Hence, the payoff to a consumer who brought z units of trees in a pairwise meeting in the DM is $u^b = u(y) + W^b(z - p) = u(y) - p + u_0^b$ where $u_0^b = W^b(z)$, as specified in Section 2. There is a similar equation defining the value function of a producer (seller of the DM goods), $W^s(z)$.

Bargaining with an agenda The terms of trade in pairwise meetings are determined according to the gradual bargaining solution described in Sections 2 and 3. An intuitive and tractable way to solve this bargaining game in a general equilibrium model is as follows. Suppose an interim agreement, (y, p) , has been reached where $y < y^*$ and the consumer adds an infinitesimal quantity ∂z of assets to the bargaining table. The outcome, $(\partial y, \partial p)$, for this new round of negotiation is given by the generalized Nash solution, i.e.,

$$(\partial y, \partial p) \in \arg \max [u'(y)\partial y - \partial p]^\theta [\partial p - v'(y)\partial y]^{1-\theta} \quad \text{s.t.} \quad \partial p \leq \partial z. \quad (30)$$

Given that the consumer has already secured a consumption level y , the surplus from the agreement $(\partial y, \partial p)$ is $u'(y)\partial y - \partial p$ where the additional amount of consumption is valued at the marginal utility, $u'(y)$. Similarly, the cost to the seller to produce an additional ∂y is $v'(y)\partial y$ and hence his surplus is $\partial p - v'(y)\partial y$. Note that the total surplus is positive as long as $y < y^*$. Provided ∂z is small, the solution to (30) is such that $\partial p = \partial z$. So the bargaining problem is the same as the one in Shi (1995) and Trejos and Wright (1995) where agents bargain over the output in exchange for an indivisible unit of money, here ∂z , according to the Nash solution. The problem is even easier in that the surpluses are linear in ∂y . It also resembles the use of the generalized Nash solution in the Lagos and Wright (2005) model except that now the negotiation takes place at the margin. The first-order condition of the maximization problem in (30) with respect to ∂y gives

$$\frac{\partial y}{\partial z} = \frac{\theta u'(y) + (1 - \theta)v'(y)}{u'(y)v'(y)}. \quad (31)$$

This solution coincides with (21) by substituting $\tau = p/\delta$. It gives the marginal value of real balances in terms of DM consumption. We can then compute the payment function, $p(y)$, by integrating $\partial p/\partial y = \partial z/\partial y$ over $[0, y]$ for all $y < y^*$, i.e.,

$$p(y) = \int_0^y \frac{u'(x)v'(x)}{\theta u'(x) + (1 - \theta)v'(x)} dx, \quad \forall y < y^*. \quad (32)$$

We denote $z^* = p(y^*)$ as the wealth required to purchase y^* . The total consumption of a buyer holding $z \leq z^*$ is then

$$y(z) = \int_0^z \frac{\theta u'(x) + (1 - \theta)v'(x)}{u'(x)v'(x)} dx. \quad (33)$$

Given the payment and consumption functions, $p(y)$ and $y(z)$, we compute the lifetime expected utility of a consumer bringing z assets to the DM:

$$V^b(z) = \alpha \{u[y(z)] + W^b\{z - p[y(z)]\}\} + (1 - \alpha) W^b(z), \quad (34)$$

According to (34) a consumer meets a producer with probability α . The consumer enjoys y units of DM consumption in exchange for p units of assets. With probability $1 - \alpha$ the consumer is unmatched and enters the CM with z units of asset.

Choice of asset holdings Substituting $V^b(z)$ with its expression given by (34), and using the linearity of $W^b(z)$, the consumer's choice of asset holdings solves

$$\max_{z \geq 0} \{-sz + \alpha \{u[y(z)] - p[y(z)]\}\}, \quad (35)$$

where s is the spread between the rate of time preference and the real rate on liquid Lucas trees,

$$s = \frac{\rho - r}{R} \geq 0. \quad (36)$$

We rewrite the portfolio problem, (35), as a choice of DM consumption, taking into account that the payment function, $p(y)$, is given by (22). It becomes:

$$\max_{y \in [0, y^*]} \left\{ -sp(y) + \alpha \int_0^y \frac{\theta u'(x) [u'(x) - v'(x)]}{\theta u'(x) + (1 - \theta)v'(x)} dx \right\}. \quad (37)$$

Note that we can restrict the choice of y to $[0, y^*]$ since the second term is maximum when $y = y^*$. The objective function is continuous and strictly concave for all $y \in (0, y^*]$. The first-order condition gives

$$s = \alpha \theta \left(\frac{u'(y)}{v'(y)} - 1 \right). \quad (38)$$

From (38) the interest rate spread has a simple expression as the product of three components: the search friction, α , the bargaining power, θ , and the marginal value of wealth in the DM, $u'(y)/v'(y) - 1$. Interestingly, the expression for the liquidity premium on the right side of (38) is much simpler than the one obtained from the Nash solution that involves the second derivatives of u and v and that is not necessarily monotone in y .

By market clearing,

$$p(y) \leq \left(\frac{1 + \rho}{\rho - s} \right) Ad, \quad " = " \quad \text{if } s > 0, \quad (39)$$

where we have used that the cum-dividend price of the asset is $\phi + d = (1 + \rho)d/(\rho - s)$. When $s > 0$, consumers hold exactly $p(y) = (\phi + d)A$. If $s = 0$, then from (38) $y = y^*$. The total supply of the asset, $(\phi + d)A$, is no less than $p(y^*)$ since assets can also be held as a pure store of value. An equilibrium can be reduced to a pair (s, y) that solves (38) and (39). We measure social welfare as the sum of surpluses in pairwise meetings, $\mathcal{W} = \alpha [u(y) - v(y)]$, but we do not include the output from Lucas trees, Ad .

Proposition 5 (*Asset prices and welfare.*) *An equilibrium exists and is unique.*

1. (**Lucas trees**, $d > 0$). *If $Ad \geq \rho p(y^*)/(1 + \rho)$ then $s = 0$ and $y = y^*$ in all matches. If $Ad < \rho p(y^*)/(1 + \rho)$ then $s > 0$ and $y < y^*$.*
2. (**Comparison to Nash.**) *Suppose $\theta < 1$. The equilibrium under Nash bargaining never implements the first best, i.e., $y < y^*$ for all $A > 0$.*
3. (**Fiat money**, $d = 0$). *For all $s > 0$, $y < y^*$. As s approaches 0, y tends to y^* for all $\theta \in (0, 1]$.*

The first part of Proposition 5 shows that the first best output in pairwise meetings is achieved for all bargaining powers provided that the asset supply is sufficiently abundant. While intuitive, the second part of Proposition 5 shows that this result does not hold under Nash bargaining. If agents bargain all at once ($N = 1$ in the repeated Rubinstein game) according to Nash, then for all $\theta < 1$, the equilibrium never achieves first best irrespective of the supply of assets. The non-monotonicity of the Nash solution generates asset misallocation by preventing the market from clearing if all the asset supply is held by consumers. As a result, a fraction of A is held by producers even though they have no liquidity needs while consumers are liquidity-constrained. This result shows that gradual bargaining is not only desirable for asset owners to increase their surpluses (Proposition 4), it is also socially desirable to avoid the misallocation of assets.

The last part of Proposition 5 is a corollary of the first part in the case of fiat money. The spread s is now taken as a policy parameter. As is standard in monetary models, as long as $s > 0$ the output is inefficiently low. However, if $s = 0$, which corresponds to the Friedman rule, then the equilibrium implements the first best for all bargaining powers. Again, it is in sharp contrast with the inability of the Friedman rule to generate the first best under Nash bargaining (Lagos and Wright, 2005).¹⁴

Using the same calibrated parameter values as Lagos and Wright (2005), $u(y) = y^{0.61}/0.61$, $v(y) = y$, and $\theta = 0.343$, we compare the output traded at the Friedman rule by playing the

¹⁴The gradual solution, $N = +\infty$, is not the only bargaining solution able to implement the first best at the Friedman rule when producers have some bargaining power. A case in point is the proportional solution proposed by Kalai (1977). See Aruoba et al. (2007). However, the Kalai solution is not scale invariant and does not have strategic foundations. Interestingly, in Hu and Rocheteau (2020) we show that in the context of quasi-linear environments the same extensive-form games we described earlier provide foundations for the proportional solution when the agenda consists in bargaining over output gradually.

game described in Section 3 for some arbitrary N relative to the first-best output, y^* , which is obtained at the limit when $N = +\infty$. Increasing N from 1 to 5 raises output in bilateral matches by about 39%, and increasing N to infinity raises it by 76%. If consumers divide their asset holdings into 5 bundles, they raise their surplus by 34%. Taking N to infinity expands their surplus by 95%.

4.3 An OTC market with linear payoffs

In order to illustrate the last part of Proposition 5, we provide a stark example of an OTC market where Nash bargaining delivers the worst possible allocation while gradual bargaining delivers the first best. We adopt a specification with linear payoffs, similar to Lagos and Zhang (2020) and consider an endowment economy. At the beginning of each period sellers (previously labeled producers) are endowed with Ω units of DM goods interpreted as short-lived assets and have a linear technology to transform each unit of the DM good into $\varepsilon_\ell > 0$ units of numéraire. Buyers (previously labeled consumers) receive no endowment but can transform the DM good into $\varepsilon_h > \varepsilon_\ell$ units of numéraire. Hence, $u(y) = \varepsilon_h y$ and $v(y) = \varepsilon_\ell y$. Sellers choose the quantity of DM goods, $\omega \leq \Omega$, to bring into a bilateral match and consume the rest.¹⁵ We set $d = 0$ so that purchases of DM goods are financed with fiat money. The spread s given by (36) is the difference between the rate of return of money, $r = -\pi/(1 + \pi)$ where π is the money growth rate implemented through lump-sum transfers, and the rate of time preference. It can also be interpreted as a nominal interest rate on an illiquid bond.

Suppose first that agents negotiate according to Nash. The outcome in a match where the buyer holds z and the seller holds ω is given by:

$$\max_{y,p} (\varepsilon_h y - p)(p - \varepsilon_\ell y) \quad \text{s.t.} \quad p \leq z \quad \text{and} \quad y \leq \omega. \quad (40)$$

If the liquidity constraint, $p \leq z$, does not bind, then the solution is $y = \omega$ and $p = (\varepsilon_h + \varepsilon_\ell)\omega/2$. Buyers purchase all the DM goods, which is socially efficient, and a payment is made to divide the match surplus evenly. This trade is feasible if $(\varepsilon_h + \varepsilon_\ell)\omega/2 \leq z$. If $p \leq z$ binds then there are two cases to distinguish. If $(\varepsilon_h + \varepsilon_\ell)z \geq 2\varepsilon_h\varepsilon_\ell\omega$, then agents

¹⁵The assumption according to which agents can choose to bring only a fraction of their asset holdings in a match was introduced by Berentsen and Rocheteau (2003), Lagos and Rocheteau (2008), and Lagos (2010). This assumption addresses the fact that under Nash bargaining agents might have incentives to hide some of their assets.

swap their inventories, $y = \omega$ and $p = z$. Otherwise, if $(\varepsilon_h + \varepsilon_\ell)z < 2\varepsilon_h\varepsilon_\ell\omega$, the buyer spends all his real balances, $p = z$, in order to purchase $y = (\varepsilon_h + \varepsilon_\ell)z / (2\varepsilon_\ell\varepsilon_h)$. The seller's surplus, $u^s(\omega, z) \equiv p(\omega, z) - \varepsilon_\ell y(\omega, z)$, is piecewise linear and non-monotone in ω . It reaches a maximum for $\omega = 2z / (\varepsilon_h + \varepsilon_\ell)$. Similarly, the buyer's surplus, $u^b(\omega, z) \equiv \varepsilon_h y(\omega, z) - p(\omega, z)$, is piecewise linear, non-monotone in z , and reaches a maximum when $z = 2\varepsilon_h\varepsilon_\ell\omega / (\varepsilon_h + \varepsilon_\ell)$.

Suppose, alternatively, that agents bargain gradually over real balances. The outcome of the negotiation for the marginal unit of money is

$$\max_{\partial y, \partial p} (\varepsilon_h \partial y - \partial p)(\partial p - \varepsilon_\ell \partial y) \quad \text{s.t.} \quad \partial p \leq \partial z \quad \text{and} \quad \partial y < \omega - y. \quad (41)$$

The first feasibility condition, $\partial p \leq \partial z$, states that the buyer cannot spend more than the ∂z units of real balances that have been added to the negotiation table. The second feasibility condition, $\partial y < \omega - y$, states that the seller cannot deliver more than his remaining inventories. As long as $\omega > y$ and provided that ∂z is infinitesimal, the constraint $\partial p \leq \partial z$ binds and the solution of (41) takes the form

$$\partial y = \left(\frac{\varepsilon_h + \varepsilon_\ell}{2\varepsilon_\ell\varepsilon_h} \right) \partial z.$$

It follows immediately that the change in the buyer's surplus is $u^b(z) = \varepsilon_h \partial y / \partial z - 1 = (\varepsilon_h - \varepsilon_\ell) / (2\varepsilon_\ell)$ if $y \leq \omega$ does not bind. The buyer's surplus is monotone increasing in his real balances. By integrating $\partial z / \partial y$ we obtain the following linear payment function,

$$p(y) = \frac{2\varepsilon_h\varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell} y. \quad (42)$$

It follows that the seller's surplus from selling ω units of DM goods, assuming the buyer has enough real balances to do so, is $p(\omega) - \varepsilon_\ell \omega = \varepsilon_\ell (\varepsilon_h - \varepsilon_\ell) \omega / (\varepsilon_h + \varepsilon_\ell)$, which is monotone increasing in ω .

Irrespective of the bargaining solution, the seller chooses $\omega \leq \Omega$ to maximize $\int u^s(\omega, z) dF^b(z)$, where $F^b(z)$ is the distribution of real balances across buyers. The buyer's problem consists in choosing z in order to maximize $-sz + \alpha \int u^b(z, \omega) dF^s(\omega)$ where $F^s(\omega)$ is the distribution of inventories held by sellers in DM matches. We characterize equilibrium allocations in the following proposition.

Proposition 6 (*Allocations in OTC markets under liquidity constraints.*) *Suppose*

sellers are endowed with Ω units of DM goods and preferences are given by $u(y) = \varepsilon_h y$ and $v(y) = \varepsilon_\ell y$. The liquid asset takes the form of fiat money, $d = 0$

1. **Nash bargaining.** For all $s \geq 0$, there exists no monetary equilibrium and the OTC market is inactive.
2. **Gradual bargaining.** If $s \leq \frac{\alpha(\varepsilon_h - \varepsilon_\ell)}{2\varepsilon_\ell}$ then there exists a monetary equilibrium implementing the first best.

Proposition 6 provides a stark illustration of the importance of the agendas of the bilateral negotiations in OTC markets. If agents bargain according to Nash, then the OTC market is inactive and money is not valued for all $s \geq 0$, even at the Friedman rule. All DM goods are held by the least productive agents, which corresponds to the worst allocation.¹⁶ We represent the seller's and buyer's best-response functions, ω^{BR} and z^{BR} , for symmetric equilibria in the left panel of Figure 9. The only intersection is when $z = \omega = 0$. If sellers bring Ω in the match, then buyers bring at most $z = 2\varepsilon_h\varepsilon_\ell\Omega/(\varepsilon_h + \varepsilon_\ell)$ real balances in order to maximize their surplus. But if sellers anticipate this amount of real balances, they will bring at most $\omega = 4\varepsilon_h\varepsilon_\ell\Omega/(\varepsilon_h + \varepsilon_\ell)^2 < \Omega$. And so on. The process unravels until neither the buyer nor the seller brings anything to trade.

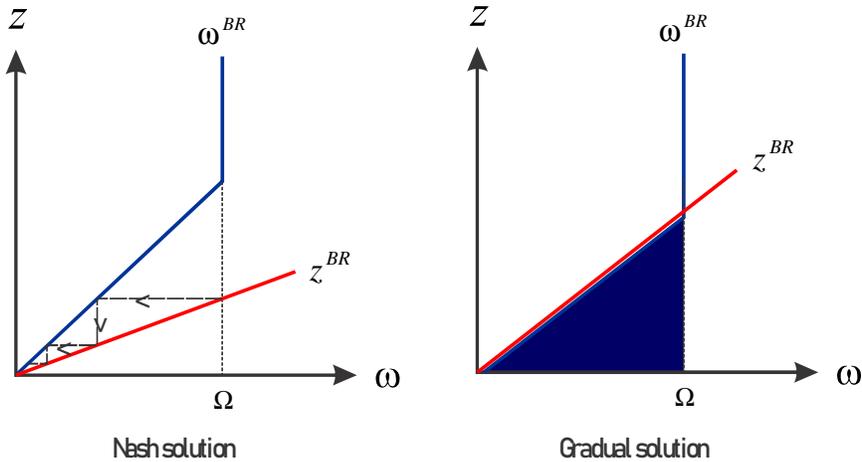


Figure 9: Symmetric best-response correspondences under Nash (left) and gradual (right) bargaining.

¹⁶This result does not rely on preferences being linear and is robust to various alternative assumptions. See Lebeau (2020) for details.

In contrast, if agents bargain gradually, then the first-best trades are implemented in all matches provided that s is not too high. We represent the best-response correspondences under gradual bargaining and assuming symmetry across agents in the right panel of Figure 9. Note that sellers bring at the minimum the amount of DM goods corresponding to what buyers can pay for and they can bring up to their full endowment Ω (i.e., their best response is an interval). For low spreads, there exists a Nash equilibrium where $\omega = \Omega$ and $z = 2\varepsilon_h\varepsilon_\ell\Omega/(\varepsilon_h + \varepsilon_\ell)$. The OTC market is active and it achieves the first best where in all matches sellers transfer all their endowments of DM goods to buyers. The unraveling that occurs under the Nash solution is avoided precisely because agents' surpluses are monotone increasing in the goods or assets they bring in a match.

5 Gradual bargaining with multiple assets

So far, we assumed that there is a single asset. We now relax this assumption and introduce multiple assets. Following Zhu and Wallace (2007), we explain rate-of-return and liquidity differences among assets from the bargaining protocol. Besides the bargaining protocol, we also need some fundamental features to distinguish assets. The feature we exploit is that it takes time to negotiate assets sequentially. In order to make this time dimension relevant, we will assume that the duration of the negotiation is stochastic and exponentially distributed. In contrast to Zhu and Wallace (2007), we do not impose an arbitrary order according to which assets are negotiated and we do not change bargaining powers across stages of the negotiation. The fundamental difference between assets will be their negotiability, δ , which is the amount of asset that can be negotiated per unit of time.

There are now J types of Lucas trees indexed by $j \in \{1, \dots, J\}$. For simplicity, we assume that the Lucas trees fully depreciate in one period for all types, and that each Lucas tree born in $t - 1$ pays off one unit of numéraire in the CM of t , i.e., $d = 1$. The supply of type- j Lucas trees is fixed at A_j and the new Lucas trees are received by consumers in a lump-sum fashion at the beginning of each CM. The CM price of Lucas tree j is ϕ_j , their gross real rate of return is $R_j = 1 + r_j = 1/\phi_j$, and the interest-rate spread relative to an illiquid asset is

$$s_j = \frac{\rho - r_j}{R_j}. \quad (43)$$

We index fiat money by $j = 0$, and hence for asset 0, $d = 0$. It is the only long-lived asset with gross real rate of return equal to $R_0 = 1/(1 + \pi)$. The spread, $s_0 = i = (1 + \rho)(1 + \pi) - 1$, is interpreted as the nominal interest rate of an illiquid asset.

In order to differentiate these assets, we take seriously the notion that it takes time to negotiate the sale of a portfolio of assets gradually over time. This notion is embodied into the concept of agenda according to which different items are negotiated sequentially. We take this sequential negotiation as a primitive, i.e., a technological constraint imposed on the negotiation. More specifically, over a small time interval of length $\Delta > 0$, agents can negotiate the sale of $\delta_j \Delta$ units of asset j (expressed in terms of numéraire), where $\delta_j > 0$ is a measure of the speed of the negotiation that captures the process of negotiating, authenticating assets, and transferring asset ownership (e.g., physical transfer, a ledger, a blockchain technology).¹⁷ We focus on the case where N is large and Δ is small. Note that here we do not think of N as a choice variable. Instead, we take gradual bargaining as a physical constraint on how assets can be traded. However, the asset owner can choose the order according to which different assets are negotiated.

We rank assets according to their negotiability, $\delta_0 \geq \delta_1 \geq \delta_2 \geq \dots \geq \delta_J$. We assume that fiat money is the most negotiable asset because it is a tangible object whose ownership is asserted by simply carrying it and it can be authenticated with relatively small effort. It takes more time to transfer and verify the ownership of non-tangible assets (e.g., cryptocurrencies), making them less negotiable. Complex financial securities take even more time to be authenticated and evaluated. In Figure 10 we provide some evidence based on Pagnotta and Philippon (2018) and O’Keeffe (2018) that transaction times vary for different classes of assets.¹⁸

Without any additional assumption, δ_j is irrelevant for the final outcome of the negotiation. In order to make time relevant, we assume that the total amount of time allocated to the negotiation, $\bar{\tau}$, is a random variable exponentially distributed with mean $1/\lambda$ and realized at the beginning of a match. The assumption of a random duration of the negoti-

¹⁷In that regard, our theory complies with the Wallace (1998) dictum in that it specifies assets by how their physical properties determine the technology to transfer their ownership, which permits the assets’ role in exchange to be endogenous.

¹⁸As mentioned earlier, it is hard to disentangle the different sources of delays in asset transactions (see, e.g., Duffie, 2012) but there is strong evidence that those delays vary across assets. In our model, we keep search frictions the same across assets and attribute all the differences to the negotiation process and the time to transfer ownership.

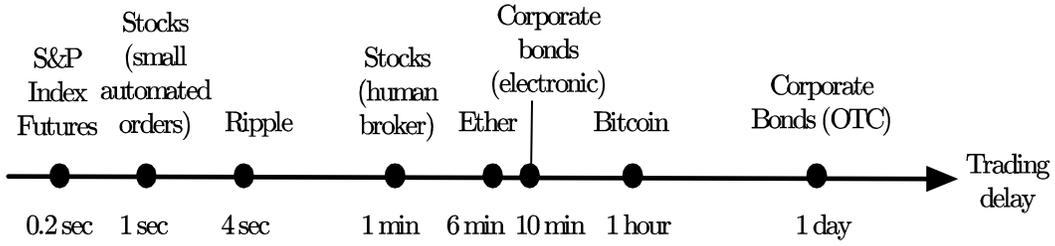


Figure 10: Trading delays by asset classes. Sources: Pagnotta and Philippon (2018), O’Keeffe (2018).

ation is commonly used in models with alternating offers (e.g., Binmore et al., 1986). The consumer’s bargaining power is θ .¹⁹

We let consumers choose the order according to which assets are sold (after $\bar{\tau}$ has been realized). The cumulative amount of asset of type j that has been up for negotiation at time τ is denoted $\omega_j(\tau)$ and $\omega(\tau) = \sum_{j=0}^J \omega_j(\tau)$ is the value of the asset portfolio that has been negotiated up to τ . It obeys the following law of motion:

$$\omega'_j(\tau) = \delta_j \sigma_j(\tau) \quad \text{for all } j \in \{0, 1, \dots, J\}, \quad (44)$$

where $\sigma_j(\tau) \in [0, 1]$ is the fraction of time devoted to the sale of asset j at time τ and $\sum_{j=0}^J \sigma_j(\tau) = 1$. So, the amount of asset j added to the negotiation table at time τ is the product of the negotiability of asset j , δ_j , and the fraction of time that the consumer dedicates to the negotiation of asset j , σ_j . Moreover, feasibility implies $\sigma_j(\tau) \in [0, 1]$ if $\omega_j(\tau) < a_j$ and $\sigma_j(\tau) = 0$ otherwise. In words, an agent can add asset j on the negotiating table at time τ only if he has not sold all his holdings of asset j prior to τ . Replacing δ by ω' in (21), the change in the consumer’s consumption over time is

$$y'(\tau) = \frac{\theta u'(y) + (1 - \theta) v'(y)}{u'(y)v'(y)} \omega'(\tau), \quad (45)$$

if $y(\tau) < y^*$ and $y'(\tau) = 0$ otherwise. The left side is the output purchased over an infinitesimal amount of time. The right side is composed of two terms: the amount of output that

¹⁹One could allow θ to be a function of τ , which would not affect our results qualitatively. One could also assume that θ varies with the type of asset that is currently under negotiation. Such extension would allow our theory to encompass the explanations for rate-of-return differences across assets by Zhu and Wallace (2007) and Rocheteau and Nosal (2017).

a marginal unit of wealth buys times the amount of wealth that can be negotiated over a small time interval.

The surplus of a consumer in a DM match with portfolio $\mathbf{a} = [a_j]_{j=0}^J$, agenda $\boldsymbol{\sigma} = [\sigma_j]_{j=0}^J$, and time to negotiate $\bar{\tau}$ is:

$$S(\mathbf{a}, \boldsymbol{\sigma}, \bar{\tau}) = \theta \int_0^{\bar{\tau}} \ell[y(\tau)] \omega'(\tau) d\tau = \theta \int_0^{\omega(\bar{\tau})} \ell[p^{-1}(\omega)] d\omega, \quad (46)$$

where $\ell(y) \equiv u'(y)/v'(y) - 1$ is the marginal surplus, and $p^{-1}(\omega) = y^*$ whenever $\omega > p(y^*)$. Over a small time interval of length $d\tau$ the consumer sells $\omega'(\tau)$ units of assets where each unit generates a marginal surplus to the consumer equal to $\theta\ell(y)$. The right side of (46) is obtained by adopting the change of variable $\omega = \omega(\tau)$. It follows that the consumer surplus depends on the agenda $\boldsymbol{\sigma}$ only through the amount of assets that can be negotiated up to $\bar{\tau}$, $\omega(\bar{\tau})$. The right side of (46) has a simple interpretation. The consumer receives a fraction θ of the sum of the marginal surpluses, $\ell(y)$, negotiated over the time interval $[0, \bar{\tau}]$. From (44),

$$\omega(\bar{\tau}) = \int_0^{\bar{\tau}} \sum_{j=0}^J \delta_j \sigma_j(\tau) d\tau.$$

The total wealth negotiated over $[0, \bar{\tau}]$ is the sum over all asset types and all infinitesimal time intervals of the marginal quantities of asset added to the negotiation table. In order to characterize the optimal strategy to maximize $\omega(\bar{\tau})$ we denote $T_0 = 0$ and

$$T_j(\mathbf{a}) = \sum_{k=0}^{j-1} \frac{a_k}{\delta_k} \text{ for all } j \in \{1, 2, \dots, J+1\}. \quad (47)$$

That is, T_j is the time that it takes to sell the first $j - 1$ most negotiable assets.

Lemma 2 (Pecking order) *For any portfolio \mathbf{a} and any realization of $\bar{\tau}$, the optimal choice $\boldsymbol{\sigma}^* = [\sigma_j^*]$ is given by*

$$\sigma_j^*(\tau) = \begin{cases} 1 & \text{if } T_j < \tau \leq T_{j+1} \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 2 shows that it is optimal to adopt a pecking order to sell assets.²⁰ Consumers start paying with money. When their money holdings are exhausted, they start selling asset

²⁰For a pecking-order theory of payments based on informational asymmetries between consumers and producers, see Rocheteau (2011).

1, etc. Hence, in a fraction $1 - e^{-\lambda T_1}$ of matches only money is used to finance consumption, where T_1 is endogenous. In a fraction $e^{-\lambda T_1} - e^{-\lambda T_2}$ of matches both money and type-1 Lucas trees serve as means of payments. And so on. Given this pecking order, the expected maximized surplus of the consumer is:

$$S(\mathbf{a}) = \int_0^{+\infty} \lambda e^{-\lambda \tau} S(\mathbf{a}, \boldsymbol{\sigma}^*, \tau) d\tau = \theta \sum_{j=0}^J \delta_j \int_{T_j}^{T_{j+1}} e^{-\lambda \tau} \ell[y(\tau)] d\tau. \quad (48)$$

Over the time interval $[T_j, T_{j+1}]$ agents negotiate asset j where the speed of the negotiation is given by δ_j .

We now turn to the asset pricing implications of this pecking order. The portfolio problem in the CM is given by

$$\max_{\mathbf{a} \geq \mathbf{0}} \{-\mathbf{s}\mathbf{a} + \alpha S(\mathbf{a})\}, \quad (49)$$

where $\mathbf{s} = [s_j]$ is the vector of asset spreads. According to (49) the consumer maximizes his expected DM surplus net of the costs of holding assets as measured by the spreads $[s_j]$. The FOCs of the maximization problem (49) are:

$$s_j = \alpha \frac{\partial S(\mathbf{a})}{\partial a_j}. \quad (50)$$

The left side of (50) is the opportunity cost of holding asset j . The right side is the probability α that the consumer receives an opportunity to spend, α , multiplied by the marginal liquidity value from holding asset j . The expression of this last term is given in the following lemma.

Lemma 3 *The marginal value of asset j to a consumer with portfolio \mathbf{a} is*

$$\frac{\partial S(\mathbf{a})}{\partial a_j} = \theta \lambda \overbrace{\sum_{k=j+1}^J \int_{T_k}^{T_{k+1}} \frac{(\delta_j - \delta_k)}{\delta_j} e^{-\lambda \tau} \ell[y(\tau)] d\tau}^{\text{negotiability value}} + \overbrace{\theta e^{-\lambda T_{J+1}} \ell[y(T_{J+1})]}^{\text{liquidity value}}. \quad (51)$$

From (51), holding an additional unit of a_j has two benefits to the consumer. First, there is a liquidity benefit according to which the consumer has more wealth, which relaxes his liquidity constraint and allows him to consume more if the negotiation is not terminated before the whole portfolio has been sold. This effect is captured by the last term on the right side and is analogous to (38). Second, there is a negotiability benefit according to which asset j speeds up the negotiation relative to less negotiable assets of types $k > j$. This first

term on the right side of (51) is asset specific, as it depends on δ_j .

By market clearing $a_j = A_j$ for all $j \geq 1$. Hence, an equilibrium can be reduced to a list $\langle a_0, \{s_j\}_{j=1}^J \rangle$ that solves (50). In the following proposition we measure the liquidity of an asset by its velocity or turnover defined as

$$\mathcal{V}_j \equiv \frac{\alpha \int_0^{+\infty} \lambda e^{-\lambda x} \int_0^x \omega'_j(\tau) 1_{\{\omega(\tau) < p(y^*)\}} d\tau dx}{A_j}. \quad (52)$$

The numerator corresponds to the aggregate quantity of asset j sold in pairwise meetings while the denominator is the supply of the asset.

In order to fix ideas, suppose there is a single asset, fiat money. From (50) and (51), y solves

$$i = \alpha \theta e^{-\lambda \frac{p(y)}{\delta_0}} \ell(y). \quad (53)$$

From (53) the nominal interest rate is equal to the product of four components: the search friction, α , the bargaining power, θ , the negotiability friction, $e^{-\lambda \frac{p(y)}{\delta_0}}$, and the marginal value of wealth in the DM, $\ell(y)$. So, bargaining frictions affect the liquidity services of money through two channels: traders' bargaining powers and the time to negotiate real balances. The negotiability term is akin to a pledgeability coefficient but it is endogenous and depends on the time it takes to negotiate assets, the stochastic time horizon of the negotiation, and the bargaining protocol as represented by $p(y)$. From (52) the velocity of money is

$$\mathcal{V}_0 = \frac{\alpha \delta_0 \left[1 - e^{-\lambda \frac{p(y)}{\delta_0}} \right]}{\lambda p(y)}.$$

As the negotiability of money tends to infinity, its velocity approaches α .

The next proposition studies the implications of our model for asset prices and liquidity in the general case with J assets.

Proposition 7 (*The negotiability structure of asset yields.*) *For all $\{A_j\}_{j=1}^J$, if $\delta_0 > \delta_1$ then there is a $\bar{i} > 0$ such that for all $i < \bar{i}$ there exists a unique steady-state monetary equilibrium with aggregate real balances $A_0(i) > 0$. Let $\Omega_1 = A_0(i)$ and for each $j = 2, \dots, J$, let $\Omega_j = A_0(i) + \sum_{k=1}^{j-1} A_k$.*

1. *If $\Omega_{j+1} < p(y^*)$ and $\delta_j > \delta_{j+1}$, then $s_j > s_{j+1}$. If $\Omega_{j+1} \geq p(y^*)$, then $s_{j+k} = 0$ for all $k \geq 0$.*

2. If $\delta_j > \delta_{j+1}$ and $p(y^*) > \Omega_j$, then $\mathcal{V}_j > \mathcal{V}_{j+1}$. If $p(y^*) \leq \Omega_j$ then $\mathcal{V}_j = 0$.
3. As λ approaches 0, $|s_j - s_{j'}|$ approaches 0 for all $j, j' \in \{0, \dots, J\}$. Asset velocity, \mathcal{V}_j , approaches α for all j such that $\Omega_j \leq p(y^*)$, 0 for all j such that $\Omega_j \geq p(y^*)$, and $\alpha [p(y^*) - \Omega_j] / A_j$ for j such that $p(y^*) \in (\Omega_j, \Omega_{j+1})$.

Proposition 7 has several implications. First, fiat money is valued for low i irrespective of the supply of Lucas trees. Even if the capitalization of all Lucas trees, $\sum_{k=1}^J A_k$, is larger than liquidity needs, $p(y^*)$, money is useful because it can be negotiated faster, thereby allowing agents to finance a larger consumption when $\bar{\tau}$ is low.

Second, even though all Lucas trees yield identical dividend streams, the equilibrium features rate-of-return differences across assets. Provided that asset supplies are not too large, assets with a high negotiability command a lower rate of return than assets with a low negotiability, i.e., $r_j < r_{j+1}$ if $\delta_j > \delta_{j+1}$. The key components of our theory is that negotiation takes time as assets are sold gradually, and not all assets can be sold at equal speed due to technological differences to authenticate and transfer assets. Part 2 of Proposition 7 shows that assets that are more negotiable have a higher velocity, which is a consequence of the endogenous pecking order. As a result, there is a positive correlation between velocity and asset prices.

Finally, Part 3 of Proposition 7 considers the limit when the expected time horizon of the negotiation becomes arbitrarily large. If the risk that the negotiation ends before the portfolio of assets has been sold goes to zero, then the rates of return of all assets converge to the same value, i.e., there is rate-of-return equality. In that case the negotiability of assets, and the order according to which they are negotiated, does not affect their rates of return. The order in which assets are sold, however, matters for velocities. Indeed, only a fraction of assets are used for transactions and those assets have a maximum velocity equal to α .

In our working paper (Rocheteau et al., 2018), we consider two applications of our model for dual asset economies. In the first application, we interpret asset 1 as a short-term government bond and study its coexistence with fiat money and the implications for open market operations. If the time horizon of the negotiation, $\bar{\tau}$, is deterministic, there is a monetary equilibrium with $T_2 = \bar{\tau}$ provided that the supply of bonds, A_1 , is not too low or too high and $\bar{\tau}$ is in some intermediate range. Output and the interest rate spread are

determined recursively according to:

$$y = p^{-1} \left[\delta_0 \bar{\tau} - \left(\frac{\delta_0 - \delta_1}{\delta_1} \right) A_1 \right] \quad (54)$$

$$s_1 = \frac{\delta_0}{\delta_1} i - \alpha \theta \left(\frac{\delta_0 - \delta_1}{\delta_1} \right) \ell(y). \quad (55)$$

Relative to the existing literature, our model generates a new effect captured by the presence of A_1 in the right side of (54), which we call negotiability effect, according to which a reduction in the supply of bonds, A_1 , reduces output and spreads. This effect requires $\delta_0 > \delta_1$. An open market sale of bonds decreases output by crowding out a highly negotiable asset, money, with a less negotiable asset, bonds. This effect, we believe, captures the common wisdom regarding the transmission of open market operations on output. When $\bar{\tau}$ is stochastic, we distinguish this negotiability effect from the standard liquidity effect in New Monetarist models according to which an increase in A_1 raises the aggregate liquidity of the economy and hence output. The negotiability effect dominates for realizations of $\bar{\tau}$ in some intermediate range while the liquidity effect dominates for large realizations of $\bar{\tau}$.

The second application studies a dual currency economy. The supply of currency 0 (e.g., the domestic currency) grows at rate π_0 and the supply of currency 1 (e.g., the foreign currency) grows at rate π_1 . Currency 0 is easier to authenticate and can be transferred faster than currency 1, i.e., $\delta_0 > \delta_1$. In the context of cryptocurrencies, currency 0 has lower confirmation times than currency 1. However, the supply of currency 0 grows faster than the supply of currency 1, $\pi_0 > \pi_1$. If $\bar{\tau}$ is in some intermediate range, there exists a unique steady-state equilibrium where both currencies 0 and 1 are valued and output solves

$$\frac{i_0 \delta_0 - i_1 \delta_1}{\delta_0 - \delta_1} = \alpha \theta \ell(y). \quad (56)$$

Inflation rates affect output according to $\partial y / \partial \pi_0 < 0$ and $\partial y / \partial \pi_1 > 0$. Moreover, currency 0 appreciates vis-a-vis currency 1 as α or θ increases or as $\bar{\tau}$ decreases. If the inflation rate of the most negotiable currency increases, then output decreases, in accordance with textbook comparative statics. However, as π_1 increases, agents find it optimal to reduce their holdings of currency 1 and raise their holdings of currency 0. As a result, they can buy more output over the time horizon $\bar{\tau}$. In the context of a dollarization equilibrium this would mean that an increase of the inflation rate of the foreign currency raises output by

reverting the dollarization process.

6 Conclusion

The objective of this paper was to introduce a new approach to bargaining into models of decentralized asset markets. More than a new solution, we advocate for a new definition of the bargaining problem for negotiations over unrestricted asset portfolios. This new definition is a natural extension of existing bargaining theories (e.g., Osborne and Rubinstein, 1990) for a new class of models of decentralized markets with richer asset holdings. It includes as a primitive the agenda of the negotiation, i.e., a partition of the portfolio into asset bundles to be sold sequentially.

Our approach complies with the Nash program: it has (multiple) strategic foundations, in the form of alternating-offer games, and axiomatic foundations. It encompasses existing bargaining solutions, such as Nash, for specific agendas. We showed through several examples that the choice of the agenda is crucial for allocations and welfare. For instance, the choice of the agenda can have dramatic implications for the functioning of OTC markets with outcomes varying from a complete break-down to the implementation of first-best trades.

In our working paper (Rocheteau et al., 2018), we provide many additional results and applications. In the companion paper of Hu and Rocheteau (2019) we show that the proportional solution of Kalai (1977) can be interpreted as a gradual solution for the agenda that consists in bargaining gradually over the output. This result is significant because it shows that one can provide strategic foundations for the Kalai solution in quasi-linear environments commonly used in search-theoretic models. In addition, while Kalai (1977) does not impose the scale invariance axiom of Nash (1950), the gradual solution is both scale invariant and ordinal. As another extension we endogenized the time it takes to negotiate assets through some costly investment before the negotiation starts. Much more can be done with this novel approach to bargaining in decentralized asset markets.

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Appendix A: Proofs of Lemmas and Propositions

The proofs of Proposition 1 and Proposition 3 can respectively be found in Appendix B and Appendix C.

Proof of Lemma 1. The Pareto frontier is derived from the program

$$u^b = \max_{y, p \geq 0} \{u(y) - p + u_0^b\} \quad \text{s.t. } p - v(y) + u_0^s \geq u^s, \quad p \leq \delta\tau.$$

The consumer chooses the terms of trade, (y, p) , to maximize his utility subject the constraint that he must guarantee some utility level u^s to the producer. If $\delta\tau \geq u^s - u_0^s + v(y^*)$, then $y = y^*$ and $p = u^s - u_0^s + v(y^*)$. Moreover, $u^b + u^s = u(y^*) - v(y^*) + u_0^b + u_0^s$. If $\delta\tau < u^s - u_0^s + v(y^*)$, then $p = \delta\tau = u^s - u_0^s + v(y)$, i.e., $y = v^{-1}(\delta\tau - u^s + u_0^s)$. ■

Proof of Proposition 2. Since $p(\tau) = \delta\tau$, (1) implies that $u^b(\tau) = u_0^b + u[y(\tau)] - \delta\tau$, and hence

$$u^{b'}(\tau) = u'(y) y'(\tau) - \delta. \quad (57)$$

The change in the consumer's utility along the gradual bargaining path the change in DM consumption as the consumer adds assets to the negotiating table, net of the asset transfer (the second term on the right side). From (15) and (57), we obtain (17). The total transfer of assets is $p(y) = \int_0^y \delta \frac{\partial \tau}{\partial x} dx$ where from (17) $\partial \tau / \partial x$ coincides with $1/y'(\tau)$ evaluated at x . ■

Proof of Proposition 4. We assume that, with no loss of generality, $z \leq p_\infty(y^*)$. This also allows us to assume that (27) has interior solutions, and, summing (27) from $n = 1$ to N :

$$\sum_{n=1}^N \left[\int_{y_{n-1}}^{y_n} \frac{v'(y_n)}{u'(y_n) + v'(y_n)} u'(x) dx + \int_{y_{n-1}}^{y_n} \frac{u'(y_n)}{u'(y_n) + v'(y_n)} v'(x) dx \right] = z.$$

It can be expressed more compactly as

$$\int_0^{y_N} \left[1 - \Theta \left(x; \frac{z}{N} \right) \right] u'(x) + \Theta \left(x; \frac{z}{N} \right) v'(x) dx = z,$$

where

$$\Theta \left(x; \frac{z}{N} \right) = \sum_{n=1}^N \frac{u'(y_n)}{u'(y_n) + v'(y_n)} 1_{(y_{n-1}, y_n]}(x)$$

and $1_{(y_{n-1}, y_n]}(x)$ is the indicator function for the interval $(y_{n-1}, y_n]$. Note that for all $N < +\infty$

and for all $x \notin \{y_n\}$,

$$\Theta\left(x; \frac{z}{N}\right) < \frac{u'(x)}{u'(x) + v'(x)}.$$

Hence,

$$\int_0^{y_N} \left[1 - \Theta\left(x; \frac{z}{N}\right)\right] u'(x) + \Theta\left(x; \frac{z}{N}\right) v'(x) dx > \int_0^{y_N} \frac{2v'(x)u'(x)}{u'(x) + v'(x)} dx.$$

So for all $N < +\infty$, the payment to finance y_N units of consumption, the left side of the inequality, is larger than the one when $N = +\infty$, the right side of the inequality. Hence, the consumer extracts the largest surplus when $N = +\infty$. ■

Proof of Proposition 5. For each $y \in (0, y^*]$, equation (38) gives a negative relationship between s and y , denoted by $s = s(y)$, with $\lim_{y \rightarrow 0} s(y) = +\infty$, and $s(y)$ is strictly decreasing. Given this function, equilibrium is given by y that satisfies (39), i.e.,

$$\left[\rho - \alpha\theta \left(\frac{u'(y) - v'(y)}{v'(y)}\right)\right] p(y) \leq (1 + \rho) Ad, \quad " = " \text{ if } y < y^* \quad (58)$$

Since the left side of (58) is strictly increasing in y from 0 when $y = \underline{y} < y^*$ solution to $\rho = \alpha\theta [u'(\underline{y}) - v'(\underline{y})] / v'(\underline{y})$ to $\rho p(y^*)$ when $y = y^*$ and the right side is constant and strictly positive, there is a unique y that satisfies (58).

Part 1. If $p(y^*) \leq (1 + \rho)Ad/\rho$, then the solution is $y = y^*$ and $s(y^*) = 0$. If $p(y^*) > (1 + \rho)Ad/\rho$, the solution is such that $y \in (\underline{y}, y^*)$ and $s > 0$.

Part 2. Suppose the terms of trade are determined according to the generalized Nash solution:

$$(y, p) \in \arg \max [u(y) - p]^\theta [p - v(y)]^{1-\theta} \quad \text{s.t. } p \leq z.$$

The first-order condition with respect to y gives

$$p = p(y) \equiv \frac{(1 - \theta)v'(y)u(y) + \theta u'(y)v(y)}{\theta u'(y) + (1 - \theta)v'(y)}.$$

For all $\theta \in (0, 1)$, it is easy to check that $u(y) - p(y)$ reaches a maximum for some $\tilde{y}_\theta < y^*$. The buyer's problem corresponds to a choice of y solution to

$$\max_{y \geq 0} \{-sp(y) + \alpha [u(y) - p(y)]\}.$$

The solution is such that $y \leq \tilde{y}_\theta < y^*$ for all $s \geq 0$ and all $\theta < 1$.

Part 3. If the asset is fiat money, then $R = (1 + \pi)^{-1}$ where π is the money growth rate and the spread is

$$s = (1 + \rho)(1 + \pi) - 1,$$

which can be interpreted as a nominal interest rate on an illiquid asset. From (38), y is the unique solution to

$$s = \alpha\theta \left(\frac{u'(y)}{v'(y)} - 1 \right).$$

If $s > 0$, $u'(y)/v'(y) > 1$ implies $y < y^*$. It is easy to check that y decreases with s because the right side is decreasing in y and as s tends to 0, y approaches y^* . ■

Proof of Proposition 6. Under Nash bargaining the seller's surplus from a trade is:

$$u^s(\omega, z) = \begin{cases} \frac{\varepsilon_h - \varepsilon_\ell}{2} \omega & \text{if } \frac{z}{\omega} \geq \frac{\varepsilon_h + \varepsilon_\ell}{2} \\ z - \varepsilon_\ell \omega & \text{if } \frac{z}{\omega} \in \left[\frac{2\varepsilon_h \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell}, \frac{\varepsilon_h + \varepsilon_\ell}{2} \right) \\ \frac{(\varepsilon_h - \varepsilon_\ell)z}{2\varepsilon_h} & \text{if } \frac{z}{\omega} < \frac{2\varepsilon_h \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell}. \end{cases} \quad (59)$$

If z/ω is sufficiently high, then all DM goods are purchased by the buyer who only spends a fraction of his real balances. In that case, the seller's surplus increases with ω . If z/ω is in some intermediate range, then the buyer can still purchase all the DM goods of the seller but he has to spend all his real balances. In this case, the seller's surplus decreases with ω . Finally, if z/ω is low, then the buyer can only purchase a fraction of the seller's DM goods, and the seller's surplus is constant. As a result, the seller's surplus reaches a maximum when $p \leq z$ starts to bind, i.e., $\omega = 2z / (\varepsilon_h + \varepsilon_\ell)$. The surplus of a buyer in a bilateral match is

$$u^b(z, \omega) = \begin{cases} \frac{\varepsilon_h - \varepsilon_\ell}{2} \omega & \text{if } \frac{z}{\omega} \geq \frac{\varepsilon_h + \varepsilon_\ell}{2} \\ \varepsilon_h \omega - z & \text{if } \frac{z}{\omega} \in \left[\frac{2\varepsilon_h \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell}, \frac{\varepsilon_h + \varepsilon_\ell}{2} \right) \\ \frac{(\varepsilon_h - \varepsilon_\ell)z}{2\varepsilon_\ell} & \text{if } \frac{z}{\omega} < \frac{2\varepsilon_h \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell}. \end{cases} \quad (60)$$

Let \bar{z} denote the highest value on the support of $F^b(z)$. Then,

$$\omega \leq \min \left\{ \frac{2\bar{z}}{\varepsilon_h + \varepsilon_\ell}, \Omega \right\}. \quad (61)$$

Let $\bar{\omega}$ denote the highest value in the support of $F^s(\omega)$. The solution is such that

$$z \leq \frac{2\varepsilon_h \varepsilon_\ell \bar{\omega}}{\varepsilon_h + \varepsilon_\ell}. \quad (62)$$

It can be checked that $(\varepsilon_h + \varepsilon_\ell)/2 > 2\varepsilon_h\varepsilon_\ell/(\varepsilon_h + \varepsilon_\ell)$, i.e., the intersection of the two best-response functions, (61) and (62), is such that the only Nash equilibrium is $\bar{z} = \bar{\omega} = 0$.

Under gradual bargaining, the Pareto frontier of the bargaining set, $u^b = \max(\varepsilon_h y - p)$ s.t. $p - \varepsilon_\ell y \geq u^s$, $p \leq z$, and $y \leq \omega$, is given by:

$$\begin{aligned} H(u^b, u^s, z, \omega) &= (\varepsilon_h - \varepsilon_\ell)\omega - u^b - u^s \text{ if } u^s \leq z - \varepsilon_\ell\omega \\ &= \frac{(\varepsilon_h - \varepsilon_\ell)z}{\varepsilon_\ell} - \frac{\varepsilon_h}{\varepsilon_\ell}u^s - u^b \text{ otherwise.} \end{aligned}$$

Hence, the gradual bargaining solution requires

$$u^{b'}(z) = -\frac{1}{2} \frac{\partial H / \partial z}{\partial H / \partial u^b} = \frac{1}{2} \frac{(\varepsilon_h - \varepsilon_\ell)}{\varepsilon_\ell}.$$

From the definition $u^{b'}(z) = \varepsilon_h \partial y / \partial z - 1$, it follows that $\partial z / \partial y = 2\varepsilon_h\varepsilon_\ell / (\varepsilon_h + \varepsilon_\ell)$. Integrating this expression, the payment function is $p(y) = \frac{2\varepsilon_h\varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell}y$. The buyer's choice of y is given by:

$$\max_{y \in [0, \omega]} \left\{ -s \frac{2\varepsilon_h\varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell}y + \alpha \left[\varepsilon_h y - \frac{2\varepsilon_h\varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell}y \right] \right\}.$$

It can be re-expressed as:

$$\max_{y \in [0, \omega]} [-s2\varepsilon_\ell + \alpha(\varepsilon_h - \varepsilon_\ell)]y.$$

Provided that $s \leq \alpha(\varepsilon_h - \varepsilon_\ell) / (2\varepsilon_\ell)$, it is optimal to choose $y = \omega$ and to hold $z = p(\omega)$.

The surplus of the seller is:

$$\begin{aligned} u^s(\omega, z) &= \min \left\{ p(\omega) - \varepsilon_\ell\omega, z - \frac{(\varepsilon_h + \varepsilon_\ell)}{2\varepsilon_h}z \right\} \\ &= \min \left\{ \varepsilon_\ell \left(\frac{\varepsilon_h - \varepsilon_\ell}{\varepsilon_h + \varepsilon_\ell} \right) \omega, \frac{\varepsilon_h - \varepsilon_\ell}{2\varepsilon_h}z \right\}. \end{aligned}$$

The seller's surplus is monotone (weakly) increasing in ω . Hence, $\omega = \Omega$ is a weakly dominant strategy. ■

Proof of Lemma 2. By (46), an optimal $[\sigma_j]_{j=0}^J$ maximizes

$$\omega(\bar{\tau}) = \sum_{j=0}^J \int_0^{\bar{\tau}} \delta_j \sigma_j(x) dx = \sum_{j=0}^J \delta_j \Delta_j,$$

where $\Delta_j \equiv \int_0^{\bar{\tau}} \sigma_j(x) dx$, subject to feasibility. We can then rewrite this problem as

$$\max_{\Delta_j, j=0, \dots, J} \sum_{j=0}^J \delta_j \Delta_j, \text{ subject to } \sum_{j=0}^J \Delta_j = \bar{\tau} \text{ and } 0 \leq \Delta_j \leq a_j / \delta_j \text{ for all } j = 0, \dots, J,$$

where the constraints follow from feasibility requirement on $[\sigma_j]_{j=0}^J$. Now, let $\tilde{j} \geq 0$ satisfy

$$\sum_{j=0}^{\tilde{j}-1} a_j / \delta_j < \bar{\tau} \leq \sum_{j=0}^{\tilde{j}} a_j / \delta_j.$$

Since $\delta_0 \geq \delta_1 \geq \dots \geq \delta_J$, it is optimal to choose $\Delta_j = a_j / \delta_j$ for all $j = 0, \dots, \tilde{j} - 1$, $\Delta_{\tilde{j}} = \bar{\tau} - \sum_{j=0}^{\tilde{j}-1} a_j / \delta_j$, and $\Delta_j = 0$ for all $j > \tilde{j}$. Hence, $[\sigma_j^*]_{j=0}^J$ restricted to $[0, \bar{\tau}]$ is optimal. It is also uniquely optimal if $\delta_0 > \delta_1 > \dots > \delta_J$. ■

Proof of Lemma 3. Define $\Omega_j(\mathbf{a}) = \sum_{k=0}^{j-1} a_k$ for all $j = 1, \dots, J+1$ with $\Omega_0(\mathbf{a}) = 0$.

We can then rewrite (48) as

$$S(\mathbf{a}) = \theta \sum_{j=0}^J \int_{\Omega_j}^{\Omega_{j+1}} e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[p^{-1}(\omega)] d\omega,$$

where $p^{-1}(\omega) = y^*$ whenever $\omega > p(y^*)$, and we have changed the variable from τ to $\omega = \omega^*(\tau)$; note that for all $\omega \in (\Omega_j, \Omega_{j+1})$,

$$\begin{aligned} (\omega^*)^{-1}(\omega) &= \frac{(\omega - \Omega_j)}{\delta_j} + T_j, \\ \frac{d}{d\omega} (\omega^*)^{-1}(\omega) &= \frac{1}{\delta_j}. \end{aligned}$$

Now, let $k \geq 0$ be given. We shall compute the derivative of $S(\mathbf{a})$ w.r.t. a_k . We will compute it by grouping the terms inside the summation into three groups: terms with $j < k$, the term with $j = k$, and terms with $j > k$. Note that $S(\mathbf{a})$ depends on a_k through terms Ω_j and T_j with $j > k$ and hence, for $j < k$,

$$\frac{\partial}{\partial a_k} \int_{\Omega_j}^{\Omega_{j+1}} e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[p^{-1}(\omega)] d\omega = 0,$$

for $j = k$,

$$\frac{\partial}{\partial a_k} \int_{\Omega_k}^{\Omega_{k+1}} e^{-\lambda \left[\frac{(\omega - \Omega_k)}{\delta_k} + T_k \right]} \ell[p^{-1}(\omega)] d\omega = -e^{-\lambda T_k} \ell[p^{-1}(\Omega_k)],$$

and for $j > k$,

$$\begin{aligned}
& \frac{\partial}{\partial a_k} \int_{\Omega_j}^{\Omega_{j+1}} e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[p^{-1}(\omega)] d\omega \\
&= -e^{-\lambda T_j} \ell[p^{-1}(\Omega_j)] + e^{-\lambda \left[\frac{(\Omega_{j+1} - \Omega_j)}{\delta_j} + T_j \right]} \ell[p^{-1}(\Omega_{j+1})] + \int_{\Omega_j}^{\Omega_{j+1}} \lambda \left[\frac{1}{\delta_j} - \frac{1}{\delta_k} \right] e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[p^{-1}(\omega)] d\omega \\
&= -e^{-\lambda T_j} \ell[p^{-1}(\Omega_j)] + e^{-\lambda T_{j+1}} \ell[p^{-1}(\Omega_{j+1})] + \int_{\Omega_j}^{\Omega_{j+1}} \lambda \left[\frac{1}{\delta_j} - \frac{1}{\delta_k} \right] e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[p^{-1}(\omega)] d\omega.
\end{aligned}$$

Thus, adding the terms up across j , we obtain

$$\frac{\partial}{\partial a_k} S(\mathbf{a}) = \theta \sum_{j=k+1}^J \int_{\Omega_j}^{\Omega_{j+1}} \lambda \left[\frac{1}{\delta_j} - \frac{1}{\delta_k} \right] e^{-\lambda \left[\frac{(\omega - \Omega_j)}{\delta_j} + T_j \right]} \ell[y(\omega)] d\omega + \theta e^{-\lambda T_{J+1}} \ell[y(\Omega_{J+1})],$$

where the terms $e^{-\lambda T_j} \ell[p^{-1}(\Omega_j)]$ cancel one another except for the very last one. Equation (51) is obtained by another change of variable back to τ . ■

Proof of Proposition 7. (1) The equilibrium is solved recursively. The FOC (50) when $j = 0$ determines a_0 . Note that, in the expression (51) for $\frac{\partial}{\partial a_0} S(\mathbf{a})$, it depends on a_0 only through T_1, \dots, T_{J+1} , and it is strictly decreasing in a_0 . Indeed, this follows directly from the fact that T_j is strictly increasing in a_0 and the difference $T_{j+1} - T_j$ is not, and that $e^{-\lambda \tau} \ell[y(\tau)]$ strictly decreases with τ . Now, the right side of (50) is also strictly positive at $a_0 = 0$ provided that $\delta_0 > \delta_1$ and equal to 0 as a_0 goes to ∞ . The threshold for the nominal interest rate below which a monetary equilibrium exists is

$$\bar{i} = \alpha \theta \lambda \sum_{k=1}^J \frac{(\delta_0 - \delta_k)}{\delta_0} \int_{T_k}^{T_{k+1}} e^{-\lambda \tau} \ell[y(\tau)] d\tau + \alpha \theta e^{-\lambda T_{J+1}} \ell[y(T_{J+1})],$$

where $T_1 = 0$, and $T_j = \sum_{k=1}^{j-1} A_k / \delta_k$ for all $j \in \{2, \dots, J+1\}$.

Given a_0 , the spreads $\{s_j\}_{j=1}^J$ are determined by (50), with $A_0 = a_0$ and $T_j = \sum_{k=0}^{j-1} A_k / \delta_k$ for all $j \in \{1, \dots, J+1\}$. From (50) we can compute the difference between two consecutive spreads:

$$s_j - s_{j+1} = \alpha \theta \lambda \frac{(\delta_j - \delta_{j+1})}{\delta_j} \int_{T_{j+1}}^{T_{j+2}} e^{-\lambda \tau} \ell[y(\tau)] d\tau.$$

Hence, $s_j - s_{j+1} > 0$ requires $\delta_j - \delta_{j+1} > 0$ and $y(T_{j+1}) < y^*$, i.e., $\Omega_{j+1} = \sum_{k=0}^j A_k < p(y^*)$.

(2) We can simplify the expression of the velocity of asset j given by (52) as

$$\begin{aligned}\mathcal{V}_j &\equiv \frac{\alpha \int_0^{+\infty} \lambda e^{-\lambda x} \int_0^x \omega_j^{*\prime}(\tau) 1_{\{\omega^*(\tau) < p(y^*)\}} d\tau dx}{A_j} = \frac{\alpha \int_0^{+\infty} e^{-\lambda \tau} \omega_j^{*\prime}(\tau) 1_{\{\omega^*(\tau) < p(y^*)\}} d\tau}{A_j} \\ &= \frac{\alpha \int_{T_j}^{T_{j+1}} e^{-\lambda \tau} \delta_j 1_{\{\omega^*(\tau) < p(y^*)\}} d\tau}{A_j},\end{aligned}$$

where the first equality changes the order of integration and the second uses the fact that $\omega_j^{*\prime}(\tau) = \delta_j 1_{\{T_j \leq \tau < T_{j+1}\}}$. Using the expressions for T_j and T_{j+1} we distinguish three cases:

$$\mathcal{V}_j = \begin{cases} A_j^{-1} \lambda^{-1} \alpha \delta_j e^{-\lambda T_j} \left(1 - e^{-\frac{\lambda}{\delta_j} A_j}\right) \\ A_j^{-1} \lambda^{-1} \alpha \delta_j e^{-\lambda T_j} \left[1 - e^{-\frac{\lambda}{\delta_j} [p(y^*) - \Omega_j]}\right] \\ 0 \end{cases} \quad \text{if } p(y^*) \begin{cases} \geq \Omega_{j+1} \\ \in (\Omega_j, \Omega_{j+1}) \\ \leq \Omega_j \end{cases} .$$

Thus, $\mathcal{V}_j > 0$ if and only if $p(y^*) > \Omega_j$. Moreover, for any j with $p(y^*) > \Omega_j$,

$$\begin{aligned}\mathcal{V}_j - \mathcal{V}_{j+1} &\geq A_j^{-1} \lambda^{-1} \alpha \delta_j e^{-\lambda T_j} \left(1 - e^{-\frac{\lambda}{\delta_j} A_j}\right) - A_{j+1}^{-1} \lambda^{-1} \alpha \delta_{j+1} e^{-\lambda T_{j+1}} \left(1 - e^{-\frac{\lambda}{\delta_{j+1}} A_{j+1}}\right) \\ &= \alpha e^{-\lambda T_{j+1}} \left[\delta_j A_j^{-1} \lambda^{-1} \left(e^{\frac{\lambda}{\delta_j} A_j} - 1\right) - \delta_{j+1} A_{j+1}^{-1} \lambda^{-1} \left(1 - e^{-\frac{\lambda}{\delta_{j+1}} A_{j+1}}\right) \right] > 0,\end{aligned}$$

where the inequality follows from the fact that

$$\frac{e^{\frac{\lambda}{\delta_j} A_j} - 1}{\frac{\lambda}{\delta_j} A_j} > 1 > \frac{1 - e^{-\frac{\lambda}{\delta_{j+1}} A_{j+1}}}{\frac{\lambda}{\delta_{j+1}} A_{j+1}}.$$

(3) It follows directly from (50) and the fact that:

$$\begin{aligned}|s_j - s_{j+1}| &= \alpha \theta \lambda \frac{(\delta_j - \delta_{j+1})}{\delta_j} \int_{T_{j+1}}^{T_{j+2}} e^{-\lambda \tau} \left[\frac{u'[y(\tau)] - v'[y(\tau)]}{v'[y(\tau)]} \right] d\tau \\ &\leq \alpha \theta \lambda \frac{(\delta_j - \delta_{j+1})}{\delta_j} e^{-\lambda T_{j+1}} \left[\frac{u'[y(T_{j+1})] - v'[y(T_{j+1})]}{v'[y(T_{j+1})]} \right] d\tau,\end{aligned} \quad (63)$$

which converges to zero as $\lambda \rightarrow \infty$. ■

Appendix B: Proof of Proposition 1 and extension to asymmetric bargaining power

As assumed in the main text, the number of bargaining rounds, N , is even, and the producer is the first to make an offer while the consumer is the last. We obtain essentially the same results for the other cases (either N is odd or the producer is making the last offer), as will be discussed in the proof. Here we also normalize $u_0^b = u_0^s = 0$. Also, with no loss of generality, we normalize δ to be one.

We define intermediate payoffs as the utilities that the players would enjoy based on the agreements reached up to some round $n \in \{1, \dots, N\}$. Let (y_n, p_n) denote the cumulative offers that are agreed upon up to round n . Feasibility requires $0 \leq p_n - p_{n-1} \leq z/N$ and $0 \leq y_n - y_{n-1}$ for all $n = 1, \dots, N$ and $p_0 = y_0 = 0$. From (1) and (2), we have $u_n^b = u(y_n) - p_n$ and $u_n^s = -v(y_n) + p_n$. The payoffs over terminal histories are simply u_N^b and u_N^s . If we restrict $y \in [0, y^*]$, then there is a one-to-one correspondence between the intermediate allocation (y, p) and the intermediate payoff (u^b, u^s) such that $H(u^b, u^s, p) = 0$.

The rest of the section consists in proving Proposition 1 followed by the extension to asymmetric bargaining powers. The proof contains four parts: the first gives a full characterization of the equilibrium payoffs of any subgame; the second gives equilibrium intermediate payoffs; the third proves uniqueness; the fourth characterizes the solution as N goes to infinity.

Final equilibrium payoffs

To solve the game, we need to solve all possible subgames. A subgame is characterized by the intermediate payoffs, denoted by (u_0^b, u_0^s) with the corresponding allocation denoted by (y_0, p_0) , and the number of rounds remaining for bargaining, denoted by J . That is, the subgame begins at round $N - J + 1$, with the intermediate payoff (u_0^b, u_0^s) that results from the bargaining in the first $N - J$ rounds. (The entire game has $(u_0^b, u_0^s) = (0, 0)$ and $J = N$.) Feasibility requires $p_0 \leq (N - J)z/N$, and we only consider $y_0 < y^*$ so that there are still gains from trade to be exploited. Our first lemma describes the final payoffs of such a game. Let $S(y) = u(y) - v(y)$ and $S^* = S(y^*)$.

Lemma 4 *Consider a game $[(u_0^b, u_0^s), J]$ with $0 \leq u_0^b + u_0^s < S^*$, and $p_0 = u[S^{-1}(u_0^b + u_0^s)] -$*

$u_0^b = u_0^s + v[S^{-1}(u_0^b + u_0^s)]$. Equilibrium final payoffs, $(\tilde{u}_j^b, \tilde{u}_j^s)$, correspond to the last term of the sequence, $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=0}^J$, defined as $(\tilde{u}_0^b, \tilde{u}_0^s) = (u_0^b, u_0^s)$, and

$$H(\tilde{u}_j^b, \tilde{u}_{j-1}^s, p_0 + jz/N) = 0 \text{ and } \tilde{u}_j^s = \tilde{u}_{j-1}^s, \text{ for } j \geq 1 \text{ odd}, \quad (64)$$

$$H(\tilde{u}_{j-1}^b, \tilde{u}_j^s, p_0 + jz/N) = 0 \text{ and } \tilde{u}_j^b = \tilde{u}_{j-1}^b, \text{ for } j \geq 2 \text{ even}. \quad (65)$$

The proof of Lemma 4 uses backward induction. When $J = 1$, the game $[(u_0^b, u_0^s), 1]$ is a standard take-it-or-leave-it offer game (with the consumer making the offer). In equilibrium, the consumer makes an offer that leaves the producer indifferent between rejecting or accepting, with the final payoff to the producer $\tilde{u}_1^s = u_0^s$. Taking this as given, the consumer spends up to z/N units of assets so that his final payoff \tilde{u}_1^b satisfies $H(\tilde{u}_1^b, u_0^s, p_0 + z/N) = 0$. (Note that the buyer will spend exactly z/N unless y^* is achieved with a slack liquidity constraint.) This proves (64) with $J = 1$.

Now consider $J = 2$, and the producer makes the first offer. If the consumer rejects the offer, the subgame becomes $[(u_0^b, u_0^s), 1]$, and the consumer can guarantee himself a final payoff of \tilde{u}_1^b , which we call the consumer's *reservation payoff*. Take this as given, the producer's offer is acceptable as long as the offer leads to a consumer final payoff no less than \tilde{u}_1^b . Thus, the producer's offer maximizes his final payoff, u_2^s , subject to $u_2^b \geq \tilde{u}_1^b$. Equivalently, the producer final payoff \tilde{u}_2^s solves $H(\tilde{u}_1^b, \tilde{u}_2^s, p_0 + 2z/N) = 0$. This proves (65) with $J = 2$. We illustrate this logic in Figure 11.

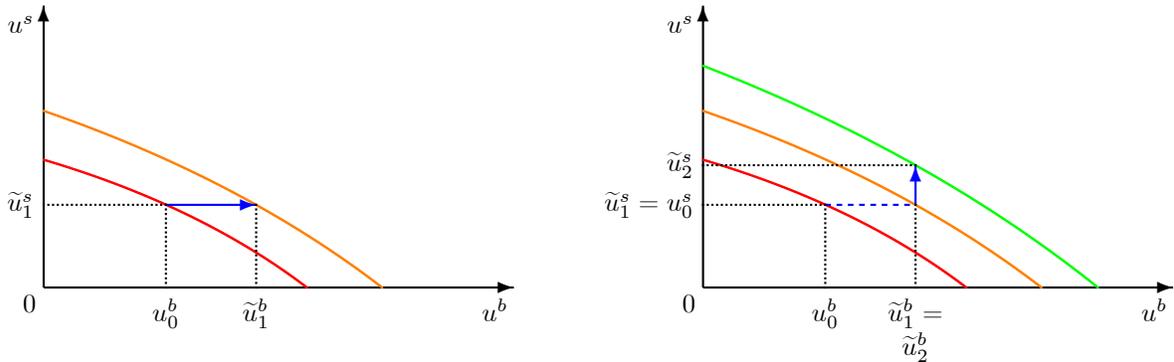


Figure 11: Construction of \tilde{u}_1^b and \tilde{u}_2^s

We continue this argument by induction. Suppose that the final payoffs are given by (64) and (65) for any game $[(u_0^b, u_0^s), J - 1]$ with $J \geq 3$ and consider a game $[(u_0^b, u_0^s), J]$

with J odd and the consumer is making the first offer. If the producer rejects the offer, his reservation payoff would be \tilde{u}_{j-1}^s . Following the same logic, the consumer's offer maximizes his final payoff u_j^b subject to the constraint that the producer's final payoff is no less than his reservation payoff, \tilde{u}_{j-1}^s . Thus, the final payoffs in the game $[(u_0^b, u_0^s), J]$, denoted by $(\tilde{u}_J^b, \tilde{u}_J^s)$, solve $H(\tilde{u}_J^b, \tilde{u}_{j-1}^s, p_0 + Jz/N) = 0$ and $\tilde{u}_J^s = \tilde{u}_{j-1}^s$. The case for J even is similar. This proves (64) and (65) for J .

Before we proceed, we give some comments on how to handle the case when the first best is reached at some point of the game. Once we reach y^* , that is, once $\tilde{u}_j^b + \tilde{u}_j^s = u(y^*) - v(y^*)$, the sequence $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=0}^J$ is constant afterwards and in equilibrium there is no trade in rounds after j . Note that this is consistent with our definition of simple SPE. Thus, we may only consider the case where

$$\tilde{u}_{j-1}^b + \tilde{u}_{j-1}^s < S^*. \quad (66)$$

Equilibrium Intermediate Payoffs

We now construct the sequence of intermediate payoffs (and the corresponding allocations and offers) that will lead to final payoffs. We emphasize that the sequence $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=0}^J$ used to construct the final payoffs is distinct from the sequence of intermediate payoffs, as we will illustrate shortly. To do so, we expand the notation slightly to explicate the recursive nature of the sequence $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=0}^J$. As mentioned, at each step according to (64)-(65), the next payoff is computed by either a rightward or upward shift to the next Pareto frontier. Formally, we define two operators, $F_r(u^b, u^s)$ and $F_u(u^b, u^s)$ given by

$$F_r(u^b, u^s) = (u^{b'}, u^{s'}) \text{ such that } u^{s'} = u^s \text{ and } H(u^{b'}, u^s, p + z/N) = 0, \quad (67)$$

$$F_u(u^b, u^s) = (u^{b'}, u^{s'}) \text{ such that } u^{b'} = u^b \text{ and } H(u^b, u^{s'}, p + z/N) = 0, \quad (68)$$

where $p = u[S^{-1}(u^b + u^s)] - u^b$. The operator $F_r(u^b, u^s)$ moves from (u^b, u^s) to the next Pareto frontier by a rightward shift, and $F_u(u^b, u^s)$ moves upward. It then follows directly from (64) and (65) that, for all j even,

$$(\tilde{u}_{j+1}^b, \tilde{u}_{j+1}^s) = F_r(\tilde{u}_j^b, \tilde{u}_j^s), \quad (69)$$

$$(\tilde{u}_{j+2}^b, \tilde{u}_{j+2}^s) = F_u(\tilde{u}_{j+1}^b, \tilde{u}_{j+1}^s) = (F_u \circ F_r)(\tilde{u}_j^b, \tilde{u}_j^s). \quad (70)$$

Our construction of equilibrium intermediate payoffs follows backward induction from the final payoffs constructed in Lemma 4. Consider a game $[(u_0^b, u_0^s), J]$ with J even. Lemma 4 shows that the final payoffs to the agents are given by $(\tilde{u}_J^b, \tilde{u}_J^s)$. Let $(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$ denote the equilibrium intermediate payoff for the agents at the end of round- $(J-1)$ bargaining. Applying Lemma 4 to the game $[(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s), 1]$, the equilibrium payoff to that game is given by $F_r(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$. Thus, subgame perfection requires

$$F_r(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s) = (\tilde{u}_J^b, \tilde{u}_J^s). \quad (71)$$

The solution to (71) is to move from $(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$ leftward to the previous Pareto frontier: formally, it is given by

$$H[\hat{u}_{J-1}^b, \hat{u}_{J-1}^s, p_0 + (J-1)z/N] = 0, \quad \hat{u}_{J-1}^s = \tilde{u}_J^s. \quad (72)$$

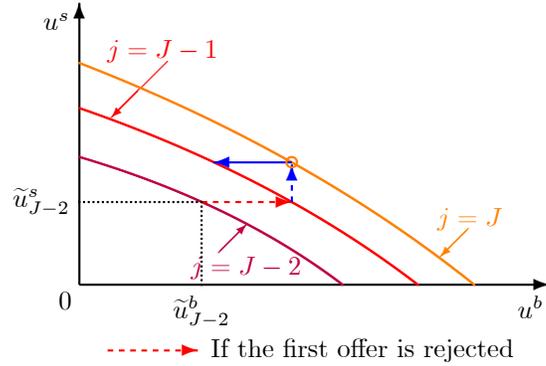


Figure 12: Backward induction

In general, the same argument shows that the equilibrium intermediate payoff at the end of round- $(J-j)$ bargaining, denoted by $(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s)$, must satisfy

$$\begin{aligned} (F_u \circ F_r)^{j/2}(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s) &= (\tilde{u}_J^b, \tilde{u}_J^s) \text{ for } j \text{ even,} \\ F_r[(F_u \circ F_r)^{(j-1)/2}(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s)] &= (\tilde{u}_J^b, \tilde{u}_J^s) \text{ for } j \text{ odd.} \end{aligned} \quad (73)$$

According to (73), for j even, if we start with $(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s)$, it should reach the final payoffs, $(\tilde{u}_J^b, \tilde{u}_J^s)$, by $j/2$ rightward and upward shifts to next Pareto frontiers, one rightward shift followed by an upward one. Now, by repeated use of (70), we have that $(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s) =$

$(\tilde{u}_{j-j}^b, \tilde{u}_{j-j}^s)$ for all j even. For j odd, if we start with $(\hat{u}_{j-j}^b, \hat{u}_{j-j}^s)$, it should reach the final payoffs, $(\tilde{u}_j^b, \tilde{u}_j^s)$, by $(j-1)/2$ rightward and upward shifts to next Pareto frontiers, plus one more rightward shift. Hence, $(\tilde{u}_{j-j}^b, \tilde{u}_{j-j}^s)$ can be obtained from $(\hat{u}_j^b, \hat{u}_j^s)$ by first a leftward shift to the previous Pareto frontier, followed by $(j-1)/2$ downward and leftward shifts to previous frontiers. Figure 12 illustrates this process for $j=2$. We have the following lemma.

Lemma 5 *Consider a game $[(u_0^b, u_0^s), J]$ be given with J even that satisfies (66). There is a unique sequence, $\{(\hat{u}_{j-j}^b, \hat{u}_{j-j}^s)\}_{j=0}^{J-1}$, with corresponding sequence of allocation denoted by $\{\hat{y}_{j-j}\}_{j=0}^{J-1}$, possibly except for $(\hat{u}_{j-1}^b, \hat{u}_{j-1}^s)$, that satisfies (73), which also enjoys the following properties:*

$$\hat{u}_{j-j}^b > \hat{u}_{j-j-1}^b \text{ for all } j = 0, \dots, J-2; \hat{u}_1^b > u_0^b; \quad (74)$$

$$\hat{u}_{j-j}^s > \hat{u}_{j-j-1}^s \text{ for all } j = 1, \dots, J-2; \hat{u}_1^s > u_0^s; \quad (75)$$

$$\hat{y}_j > \hat{y}_{j-1} \text{ for all } j = 2, \dots, J; \hat{y}_1 > y_0. \quad (76)$$

The proof of Lemma 5 is based on induction on j and uses the fact that $u(y) - v(y)$ is strictly concave. The proof is rather straightforward but tedious and the detailed proof is available upon request. Moreover, since $(\hat{u}_{j-j}^b, \hat{u}_{j-j}^s) = (\tilde{u}_{j-j}^b, \tilde{u}_{j-j}^s)$ for all j even, (74) and (75) imply that the two sequences, $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=1}^{J-1}$ and $\{(\hat{u}_{j-j}^b, \hat{u}_{j-j}^s)\}_{j=1}^{J-1}$ in fact nests one another, and hence, if one sequence converges to some limit, the other also converges to the same limit. We also remark that while we have assumed J to be even, an analogous lemma for J odd holds as well. In that case, $(\hat{u}_{j-j}^b, \hat{u}_{j-j}^s) = (\tilde{u}_{j-j}^b, \tilde{u}_{j-j}^s)$ for all j odd, but we need to compute $(\hat{u}_{j-j}^b, \hat{u}_{j-j}^s)$ for j even with an alternative sequence analogous to the one we constructed for the case with J even and j odd.

Uniqueness of SPE

Here we prove our uniqueness claim. First we show that, for any subgame, $[(u_0^b, u_0^s), J]$, the equilibrium final payoffs in any SPE is given by (64)-(65), denoted by $(\tilde{u}_j^b, \tilde{u}_j^s)$. For $J=1$ this is the standard ultimatum game and the uniqueness is standard. Suppose that we have uniqueness for $J-1$, $J \geq 2$. Then, fix a SPE and consider the game at the first bargaining round, and, without loss of generality, assume that producer is making an offer and J is even. We show that the consumer can guarantee a final payoff of \tilde{u}_j^b and the producer can guarantee

\tilde{u}_j^s at the first round. First, by rejecting the producer offer, by the induction hypothesis, the unique equilibrium payoff to the consumer is $\tilde{u}_{j-1}^b = \tilde{u}_j^b$, and hence any offer that leads to a final payoff lower than \tilde{u}_j^b will be rejected. For the consumer, Lemma 5 shows that there exists a unique intermediate payoff, (u_1^b, u_1^s) , such that $F_r \circ (F_u \circ F_r)^{(J-2)/2}(u_1^b, u_1^s) = (\tilde{u}_j^b, \tilde{u}_j^s)$, and such intermediate payoff is achievable with some offer (y_1, d_1) . By offering $(y_1 + \varepsilon, d_1)$ for ε small the producer can guarantee consumer acceptance and hence, taking ε to zero, the producer can guarantee a final payoff of \tilde{u}_j^s . Since the payoffs, $(\tilde{u}_j^b, \tilde{u}_j^s)$, lie on the Pareto frontier achievable by the two agents with total assets the consumer has, and each can guarantee the respective payoff, this final payoff is unique.

Now we show that the intermediate payoffs we constructed are unique in a simple SPE. Note first that in a simple SPE, the game effectively ends when active rounds end. Let J be the number of active rounds and the final payoffs are given by $(\tilde{u}_J^b, \tilde{u}_J^s)$, and, by backward induction, (73) must hold. Lemma 5 implies that there is a unique solution to that except for $(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$. However, that payoff can be pinned down by the fact that buyer has to spend z/N in a simple SPE in round $J - 1$. Finally, when the output corresponding to $(\tilde{u}_J^b, \tilde{u}_J^s)$ is less than y^* , then $J = N$, and the solution to (73) is unique for all j . Since y^* is not achievable in any subgame, it follows that the SPE is unique.

Convergence to Gradual Nash Solution

We consider convergence of games with N even. The limit will be the same for N odd and hence we have convergence. Here we show that the limit intermediate payoffs converge as N approaches infinity in simple SPE in the following sense. Now, for each N and each $n \in \{1, 2, \dots, N\}$, define

$$[u_N^b(\tau), u_N^s(\tau)] = (u_n^b, u_n^s) \text{ if } \tau \in [(n-1)z/N, nz/N),$$

where (u_n^b, u_n^s) is an equilibrium intermediate payoff in the game with N rounds. We then show that $[u_N^b(\tau), u_N^s(\tau)]$ converges (pointwise) to $[u^b(\tau), u^s(\tau)]$, the solution to (6) and (7).

As we have seen, the sequence of intermediate equilibrium payoffs, $\{(\hat{u}_n^b, \hat{u}_n^s)\}_{n=1}^N$, satisfies $(\hat{u}_n^b, \hat{u}_n^s) = (\tilde{u}_n^b, \tilde{u}_n^s)$ for n even. Consider two bargaining rounds, $n-1$ and $n+1$, where n is an odd number. So, $(\tilde{u}_{n-1}^b, \tilde{u}_{n-1}^s)$ and $(\tilde{u}_{n+1}^b, \tilde{u}_{n+1}^s)$ are corresponding equilibrium intermediate payoffs.

Fix some τ and let $nz/N \rightarrow \tau$ as N goes to infinity. Let $\Delta u^b = \tilde{u}_{n+1}^b - \tilde{u}_{n-1}^b$ (note that $\tilde{u}_{n+1}^b = \tilde{u}_n^b$) denote the buyer's incremental payoffs (on the equilibrium path) in rounds $n-1$ and $n+1$, and $\Delta u^s = \tilde{u}_{n+1}^s - \tilde{u}_{n-1}^s$ (note that $\tilde{u}_n^s = \tilde{u}_{n-1}^s$) denote the producer's incremental payoff (on the equilibrium path) in rounds $n-1$ and $n+1$. Similarly, let $\Delta z = 2z/N$. Then we have

$$H(\tilde{u}_{n-1}^b, \tilde{u}_{n-1}^s; \frac{n-1}{N}z) = 0 \quad (77)$$

$$H(\tilde{u}_{n-1}^b + \Delta u^b, \tilde{u}_{n-1}^s; \frac{n-1}{N}z + \frac{\Delta z}{2}) = 0 \quad (78)$$

$$H(\tilde{u}_{n-1}^b + \Delta u^b, \tilde{u}_{n-1}^s + \Delta u^s; \frac{n-1}{N}z + \Delta z) = 0. \quad (79)$$

According to (77) and (78), the producer's intermediate payoff is unchanged at \tilde{u}_{n-1}^s while the consumer's intermediate payoff increases by Δu^b . The amount of assets up for negotiation on the n^{th} frontier are nz/N . According to (79), at the end of round $n+1$ the intermediate payoffs are obtained by moving vertically from the n^{th} frontier to the $(n+1)^{\text{th}}$ frontier (since $n+1$ is even).

A first-order Taylor series expansion of (78) in the neighborhood of $(u^b, u^s, \tau) = (\tilde{u}_{n-1}^b, \tilde{u}_{n-1}^s, \frac{n-1}{N}z)$ yields:

$$H(\tilde{u}_{n-1}^b + \Delta u^b, \tilde{u}_{n-1}^s; \frac{n}{N}z) = H_1 \Delta u^b + H_3 \frac{\Delta z}{2} + o(\Delta u^b) + o(\frac{1}{N}),$$

where $\lim_{N \rightarrow \infty, nz/N \rightarrow \tau} \frac{o(\Delta u^b)}{\Delta u^b} = \lim_{N \rightarrow \infty} No(\frac{1}{N}) = 0$, we used that $H(\tilde{u}_{n-1}^b, \tilde{u}_{n-1}^s; \frac{n-1}{N}z) = 0$ from (77), and the partial derivatives H_1 , H_2 , and H_3 are evaluated at $(\tilde{u}_{n-1}^b, \tilde{u}_{n-1}^s, \frac{n-1}{N}z)$. Similarly, a first-order Taylor series expansion of (79) yields

$$H(\tilde{u}_{n-1}^b + \Delta u^b, \tilde{u}_{n-1}^s + \Delta u^s; \frac{n+1}{N}z) = H_1 \Delta u^b + H_2 \Delta u^s + H_3 \Delta z + o(\Delta u^b) + o(\Delta u^s) + o(\frac{1}{N}),$$

where $\lim_{N \rightarrow \infty, nz/N \rightarrow \tau} \frac{o(\Delta u^b)}{\Delta u^b} = \lim_{N \rightarrow \infty, nz/N \rightarrow \tau} \frac{o(\Delta u^s)}{\Delta u^s} = \lim_{N \rightarrow \infty} No(\frac{1}{N}) = 0$. Using that $H = 0$ for payoffs on the Pareto frontiers, we obtain that

$$\begin{aligned} H_1 \Delta u^b + o(\Delta u^b) &= -H_3 \frac{\Delta z}{2} + o(\frac{1}{N}), \\ H_1 \Delta u^b + o(\Delta u^b) + H_2 \Delta u^s + o(\Delta u^s) &= -H_3 \Delta z + o(\frac{1}{N}), \\ o(\Delta u^b) + H_2 \Delta u^s + o(\Delta u^s) &= -H_3 \frac{\Delta z}{2} + o(\frac{1}{N}). \end{aligned}$$

From the first one and rearranging terms, we obtain

$$\frac{\Delta u^b}{\Delta z} = -\frac{H_3}{2H_1} + \frac{o(\Delta u^b)}{H_1 \Delta z} + \frac{o(\frac{1}{N})}{H_1 \Delta z}.$$

Note that

$$\lim_{N \rightarrow \infty} \frac{o(\frac{1}{N})}{H_1 \Delta z} = \frac{o(\frac{1}{N})N}{H_1 z} = 0 \text{ and } \lim_{N \rightarrow \infty} \frac{o(\Delta u^b)}{H_1 \Delta z} = \lim_{N \rightarrow \infty} \frac{o(\Delta u^b)}{H_1 z \Delta u^b} (\Delta u^b N) = 0,$$

where

$$\Delta u^b N = (\tilde{u}_{n+1}^b - \tilde{u}_{n-1}^b)N \in [[1 - v'(\tilde{y}_{n+1})/u'(\tilde{y}_{n+1})]z, [1 - v'(\tilde{y}_{n-1})/u'(\tilde{y}_{n-1})]z]$$

and hence its limit exists and is bounded away from zero by the concavity of the function $S(\bullet)$. Thus,

$$\frac{\partial u^b}{\partial \tau} = \lim_{N \rightarrow \infty} \frac{\Delta u^b}{\Delta z} = -\frac{1}{2} \frac{H_3}{H_1} = -\frac{1}{2} \frac{\partial H / \partial \tau}{\partial H / \partial u^b}.$$

Similarly, combining these two equations and rearranging, we obtain

$$\frac{\Delta u^s}{\Delta z} = -\frac{H_3}{2H_2} + \frac{o(\Delta u^b)}{H_2 \Delta z} + \frac{o(\Delta u^s)}{H_2 \Delta z} + \frac{o(\frac{1}{N})}{H_2 \Delta z}.$$

By the same arguments, we have

$$\frac{\partial u^s}{\partial \tau} = \lim_{N \rightarrow \infty} \frac{\Delta u^s}{\Delta z} = -\frac{1}{2} \frac{H_3}{H_2} = -\frac{1}{2} \frac{\partial H / \partial \tau}{\partial H / \partial u^s}.$$

These correspond to (6) and (7).

Asymmetric bargaining powers

Here we revise our game to support gradual Nash solution with asymmetric bargaining power, denoted by θ . The parameter θ affects the game as follows. We assume that the number of rounds is an even number N , and the producer is the one making the first offer and the consumer is making the last offer.

1. In each round $n \in \{1, 3, \dots, N-1\}$, it is the producer's turn to make an offer, with asset transfer within the range $[0, 2(1-\theta)z/N]$; the consumer then decides to accept or reject the offer.

2. In each round $n \in \{2, 4, \dots, N\}$, it is the consumer's turn to make an offer, with asset transfer within the range $[0, 2\theta z/N]$; the producer then decides to accept or reject the offer.

Note that at the end of an odd round n , the maximum cumulative asset transfer is $[2(n-1) + 2(1-\theta)]z/N$, and at the end of an even round n , the maximum cumulative asset transfer is nz/N .

As before, to solve the game, we need to solve all possible subgames. Also, such subgame can still be characterized by $[(u_0^b, u_0^s), J]$, where (u_0^b, u_0^s) is the intermediate payoff at the beginning of the subgame and J is the number of remaining bargaining rounds.

Proposition 8 *Fix some $\theta \in [0, 1]$. There exists a SPE in each alternating-ultimatum offer game, and all SPE share the same final payoffs. When the output level corresponding to the final payoffs is less than y^* , the SPE is unique and is simple; otherwise, there is a unique simple SPE. Moreover, in any simple SPE, the intermediate payoffs, $\{(u_n^b, u_n^s)\}_{n=1,2,\dots,N}$, converge to the solution $[u^b(\tau), u^s(\tau)]$ to the differential equations (19) and (20) as N approaches ∞ and $[(n-1) + 2(1-\theta)]z/N$ or nz/N approaches τ .*

Note that Proposition 1 is a special case of Proposition 8 with $\theta = 1/2$. The proof of Proposition 8 follows exactly the same outline as that of Proposition 1. In particular, we will use the same technique to compute the final payoffs for any subgame, but with necessary modification to accommodate the fact that the consumer has control over θ fraction of assets to be negotiated every two rounds. As before, we can denote an arbitrary subgame by $[(u_0^b, u_0^s), J]$ with $0 \leq u_0^b + u_0^s < u(y^*) - v(y^*)$.

The final payoff is computed as follows. Define $\{(\tilde{u}_j^b, \tilde{u}_j^s)\}_{j=0}^J$ as $(\tilde{u}_0^b, \tilde{u}_0^s) = (u_0^b, u_0^s)$, and

$$H(\tilde{u}_j^b, \tilde{u}_{j-1}^s, p_0 + 2\theta z/N + (j-1)z/N) = 0, \text{ and } \tilde{u}_j^s = \tilde{u}_{j-1}^s, \text{ for } j \geq 1 \text{ odd}, \quad (80)$$

$$H(\tilde{u}_{j-1}^b, \tilde{u}_j^s, p_0 + jz/N) = 0, \text{ and } \tilde{u}_j^b = \tilde{u}_{j-1}^b, \text{ for } j \geq 2 \text{ even}, \quad (81)$$

where $p_0 = u[S^{-1}(u_0^b + u_0^s)] - u_0^b$. Below we show that the final equilibrium payoffs for the agents are given by $(\tilde{u}_J^b, \tilde{u}_J^s)$.

The logic behind this construction is exactly the same as the symmetric case, except for the fact that the consumer and the producer controls different shares of assets up for

negotiation. In particular, when $J = 1$, the game $[(u_0^b, u_0^s), 1]$ is a standard take-it-or-leave-it offer game (with the consumer making the offer). Since the consumer can offer up to additional $2\theta z/N$ units of assets, the final payoff is computed by a rightward shift to next Pareto frontier with intermediate payments $p_0 + 2\theta z/N$, as in (80) with $j = 0$. When $J = 2$, the producer makes the first offer and take the final payoff for consumer in case he rejects the offer as given. Note that with $J = 2$ the final Pareto frontier has intermediate payment of $p_0 + 2z/N$, as in (81) with $j = 0$.

To compute the intermediate payoffs, we first define the functions F_r and F_u analogous to (67) and (68):

$$F_r(u^b, u^s) = (u^{b'}, u^{s'}) \text{ such that } u^{s'} = u^s \text{ and } H(u^{b'}, u^s, p + 2\theta z/N), \quad (82)$$

$$F_u(u^b, u^s) = (u^{b'}, u^{s'}) \text{ such that } u^{b'} = u^b \text{ and } H(u^b, u^{s'}, p + 2(1 - \theta)z/N), \quad (83)$$

where $p = u[S^{-1}(u^b + u^s)] - u^b$. Now we are ready to explain how to compute intermediate payoffs. Consider a game $[(u_0^b, u_0^s), J]$ with J even. Using the same backward induction argument as in the symmetric case, if $(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$ is the equilibrium intermediate payoff for the agents at the end of round- $(J - 1)$ bargaining, then

$$F_r(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s) = (\hat{u}_J^b, \hat{u}_J^s). \quad (84)$$

As before, the solution would be obtained by a leftward shift, but, under θ , to the lower Pareto frontier with intermediate payment lowered by $2\theta z/N$; that is,

$$H(\hat{u}_{J-1}^b, \hat{u}_J^s, p_0 + Jz/N - 2\theta z/N) = 0, \quad \hat{u}_{J-1}^s = \hat{u}_J^s. \quad (85)$$

Note that in this case, $(\hat{u}_{J-1}^b, \hat{u}_{J-1}^s)$ and $(\hat{u}_{J-1}^b, \hat{u}_J^s)$ do not lie on the same Pareto frontier unless $\theta = 1/2$.

In general, we can still use (73) to compute the equilibrium intermediate payoff at the end of round- $(J - j)$ bargaining, denoted by $(\hat{u}_{J-j}^b, \hat{u}_{J-j}^s)$, with F_r and F_u defined by (82)-(83), and we have an analogous result to that of Lemma 5 for the existence and uniqueness of such a sequence. For j even the terms are obtained as before. For j odd, we need a second

sequence, $\{(\bar{u}_{J-j}^b, \bar{u}_{J-j}^s)\}_{j=0}^{J-1}$ as follows: $(\bar{u}_J^b, \bar{u}_J^s) = (\tilde{u}_J^b, \tilde{u}_J^s)$, and

$$H(\bar{u}_{J-j}^b, \bar{u}_{J-j+1}^s, p_0 + (J-j-1)z/N + 2(1-\theta)z/N) = 0, \quad (86)$$

$$\text{and } \bar{u}_{J-j}^s = \bar{u}_{J-j+1}^s \text{ for } j \geq 1 \text{ odd,}$$

$$H(\bar{u}_{J-j+1}^b, \bar{u}_{J-j}^s, p_0 + (J-j)z/N) = 0, \quad (87)$$

$$\text{and } \bar{u}_{J-j}^b = \bar{u}_{J-j+1}^b \text{ for } j \geq 1 \text{ even.}$$

Graphically, for j odd, $(\bar{u}_{J-j}^b, \bar{u}_{J-j}^s)$ is obtained from $(\bar{u}_{J-j+1}^b, \bar{u}_{J-j+1}^s)$ by moving toward left to the next lower Pareto frontier, with a decrease of incremental transfer of $2\theta z/N$; for j even, $(\bar{u}_{J-j}^b, \bar{u}_{J-j}^s)$ is obtained from $(\bar{u}_{J-j+1}^b, \bar{u}_{J-j+1}^s)$ by moving downward to the next lower Pareto frontier, with a decrease of incremental transfer of $2(1-\theta)z/N$. Note that $(\bar{u}_{J-1}^b, \bar{u}_{J-1}^s) = (\tilde{u}_{J-1}^b, \tilde{u}_{J-1}^s)$ given by (85). Note also that, in contrast to the symmetric case, $(\tilde{u}_{J-j}^b, \tilde{u}_{J-j}^s)$ is situated in the same Pareto frontier as $(\bar{u}_{J-j}^b, \bar{u}_{J-j}^s)$ if and only if j is even; for j odd, $(\bar{u}_{J-j}^b, \bar{u}_{J-j}^s)$ lies on a different frontier.

Now we show that the intermediate payoffs converge to the same limit. As in the symmetric case, consider convergence of games with N even. The limit will be the same for N odd and hence we have convergence. By the above arguments we have that the sequence of intermediate equilibrium payoffs at the end of each round is given by $\{(\hat{u}_n^b, \hat{u}_n^s)\}_{n=1}^N$ with $(u_0^b, u_0^s) = (0, 0)$, and that $(\hat{u}_n^b, \hat{u}_n^s) = (\tilde{u}_n^b, \tilde{u}_n^s)$ for n even. Consider two bargaining rounds, n and $n+2$ with n even. So, $(\tilde{u}_n^b, \tilde{u}_n^s)$ and $(\tilde{u}_{n+2}^b, \tilde{u}_{n+2}^s)$ are corresponding equilibrium intermediate payoffs. Let $\Delta u^b = \tilde{u}_{n+2}^b - \tilde{u}_n^b$ denote the buyer's incremental payoffs (on the equilibrium path) in rounds n and $n+2$, and $\Delta u^s = \tilde{u}_{n+2}^s - \tilde{u}_n^s$ denote the producer's incremental payoff (on the equilibrium path) in rounds n and $n+2$. Let $\Delta z = 2z/N$ be the corresponding change in assets. Then we have

$$H(\tilde{u}_n^b, \tilde{u}_n^s; nz/N) = 0 \quad (88)$$

$$H(\tilde{u}_n^b + \Delta u^b, \tilde{u}_n^s; \theta \Delta z + \frac{n}{N}z) = 0 \quad (89)$$

$$H(\tilde{u}_n^b + \Delta u^b, \tilde{u}_n^s + \Delta u^s; nz/N + \Delta z) = 0. \quad (90)$$

A first-order Taylor series expansion of (89) in the neighborhood of $(u^b, u^s, \tau) = (\tilde{u}_n^b, \tilde{u}_n^s, \frac{n}{N}z)$ yields:

$$H(\tilde{u}_n^b + \Delta u^b, \tilde{u}_n^s; \theta \Delta z + \frac{n}{N}z) = H_1 \Delta u^b + H_3 \theta \Delta z + o(\Delta u^b) + o(\frac{1}{N}),$$

where $\lim_{N \rightarrow \infty, nz/N \rightarrow \tau} \frac{o(\Delta u^b)}{\Delta u^b} = \lim_{N \rightarrow \infty} No(\frac{1}{N}) = 0$, we used that $H(\tilde{u}_n^b, \tilde{u}_n^s; \frac{n}{N}z) = 0$ from (88), and the partial derivatives H_1 , H_2 , and H_3 are evaluated at $(\tilde{u}_n^b, \tilde{u}_n^s, \frac{n}{N}z)$. Similarly, a first-order Taylor series expansion of (90) yields

$$H(\tilde{u}_n^b + \Delta u^b, \tilde{u}_n^s + \Delta u^s; \frac{n+2}{N}z) = H_1 \Delta u^b + H_2 \Delta u^s + H_3 \Delta z + o(\Delta u^b) + o(\Delta u^s) + o(\frac{1}{N}),$$

where $\lim_{N \rightarrow \infty, nz/N \rightarrow \tau} \frac{o(\Delta u^b)}{\Delta u^b} = \lim_{N \rightarrow \infty, nz/N \rightarrow \tau} \frac{o(\Delta u^s)}{\Delta u^s} = \lim_{N \rightarrow \infty} No(\frac{1}{N}) = 0$. Using that $H = 0$ for payoffs on the Pareto frontiers, we obtain that

$$\begin{aligned} H_1 \Delta u^b + o(\Delta u^b) &= -H_3 \theta \Delta z + o(\frac{1}{N}), \\ H_1 \Delta u^b + o(\Delta u^b) + H_2 \Delta u^s + o(\Delta u^s) &= -H_3 \Delta z + o(\frac{1}{N}), \\ H_2 \Delta u^s + o(\Delta u^s) + o(\Delta u^b) &= -(1-\theta)H_3 \Delta z + o(\frac{1}{N}) \end{aligned}$$

From the first equation with rearranging, we obtain

$$\frac{\Delta u^b}{\Delta z} = -\theta \frac{H_3}{H_1} + \frac{o(\Delta u^b)}{H_1 \Delta z} + \frac{o(\frac{1}{N})}{H_1 \Delta z}.$$

Similarly, from the third equation with rearranging, we obtain

$$\frac{\Delta u^s}{\Delta z} = -(1-\theta) \frac{H_3}{H_2} + \frac{o(\Delta u^b)}{H_2 \Delta z} + \frac{o(\Delta u^s)}{H_2 \Delta z} + \frac{o(\frac{1}{N})}{H_2 \Delta z}.$$

Thus, we have

$$\begin{aligned} \frac{\partial u^b}{\partial \tau} &= \lim_{N \rightarrow \infty, 2n/N \rightarrow \tau} \frac{\Delta u^b}{\Delta z} = -\theta \frac{H_3}{H_1} = -\theta \frac{\partial H / \partial \tau}{\partial H / \partial u^b}, \\ \frac{\partial u^s}{\partial \tau} &= \lim_{N \rightarrow \infty, 2n/N \rightarrow \tau} \frac{\Delta u^s}{\Delta z} = -(1-\theta) \frac{H_3}{H_2} = -(1-\theta) \frac{\partial H / \partial \tau}{\partial H / \partial u^s}. \end{aligned}$$

Appendix C: Proof of Proposition 3

We use backward induction to prove Proposition 3.

Round N

Consider the alternating-offer game in the last round, N . The cumulative offer up to round N is (y_{N-1}, d_{N-1}) with associated payoff (u_{N-1}^b, u_{N-1}^s) . So, if no agreement is reached in round

N , the terminal payoffs are (u_{N-1}^b, u_{N-1}^s) . The maximum wealth that can be negotiated at the end of round N is $z_N = d_{N-1} + z/N$. We will show that at the limit, when ξ_N goes to 1 (the risk of breakdown vanishes), the unique SPE payoffs of the subgame starting at the beginning of round N are determined according to the symmetric Nash solution:

$$\max_{u_N^b, u_N^s} (u_N^b - u_{N-1}^b) (u_N^s - u_{N-1}^s) \quad \text{s.t.} \quad H(u_N^b, u_N^s, z_N) = 0. \quad (91)$$

The terminal payoffs maximize the Nash product subject to the constraint that they belong to the Pareto frontier associated with z_N units of wealth. The ratio of the first-order conditions give

$$\frac{u_N^b - u_{N-1}^b}{u_N^s - u_{N-1}^s} = \frac{H_2(u_N^b, u_N^s, z_N)}{H_1(u_N^b, u_N^s, z_N)}. \quad (92)$$

At the optimum the slope of the Nash product is equal to the slope of the Pareto frontier.

We focus on the existence of the SPE and its construction. For simplicity we assume that y^* is never achieved. For the proof of uniqueness, see Rubinstein (1982). When it is his turn to make an offer the consumer proposes the (cumulative) offer (y^b, d^b) and the producer proposes (y^s, d^s) . The consumer and the producer have a reservation surplus to accept offers, u^b and u^s , respectively, which are determined endogenously below. The consumer's offer solves:

$$u_N^b = \max_{y^b, d^b} \{u(y^b) - d^b\} \quad \text{s.t.} \quad -v(y^b) + d^b \geq u^s \quad \text{and} \quad d^b \leq z_N. \quad (93)$$

The consumer maximizes his surplus subject to the constraint that his offer must generate a surplus for the producer that is at least equal to u^s and the offer must be feasible, $d^b \leq z_N$. Hence, u_N^b satisfies $H(u_N^b, u^s, z_N) = 0$. A solution to (93) exists provided that $u(y) - v(y) \geq u^s$ where $y = \min\{u^{-1}(z_N), y^*\}$. The reservation surplus of the producer solves

$$u^s = (1 - \xi_N)u_{N-1}^s + \xi_N [-v(y^s) + d^s]. \quad (94)$$

If the producer rejects the offer, his expected utility is equal to the weighted average of u_{N-1}^s , if the negotiation ends, and $-v(y^s) + d^s$ if the producer has the opportunity to make a counter-offer. Similarly, the producer's offer solves:

$$u_N^s = \max_{y^s, d^s} \{-v(y^s) - d^s\} \quad \text{s.t.} \quad u(y^s) - d^s = u^b \quad \text{and} \quad d^s \leq z_N, \quad (95)$$

where the reservation surplus of the consumer solves:

$$u^b = (1 - \xi_N)u_{N-1}^b + \xi_N [u(y^b) - d^b]. \quad (96)$$

Hence, u_N^s satisfies $H(u^b, u_N^s, z_N) = 0$. A solution to (95) exists provided that $u(y) - v(y) \geq u^b$ where $y = \min\{v^{-1}(z_N), y^*\}$. Substituting u^b and u^s by their expressions given by (94) and (96), the equilibrium payoffs, (u_N^b, u_N^s) , solve the following system of equations:

$$H [u_N^b, (1 - \xi_N)u_{N-1}^s + \xi_N u_N^s, z_N] = 0, \quad (97)$$

$$H [(1 - \xi_N)u_{N-1}^b + \xi_N u_N^b, u_N^s, z_N] = 0. \quad (98)$$

It is standard to check that for all $\xi_N < 1$ this system admits a unique solution. See Figure 13. By virtue of the one-stage-deviation principle, the proposed strategies form a SPE.

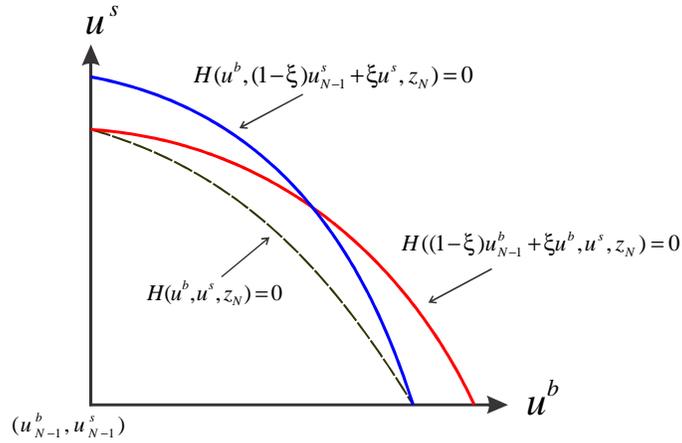


Figure 13: Determination of equilibrium payoffs

Let us consider the limit as ξ_N approaches to 1. Using a first-order Taylor series expansion we can rewrite (97)-(98) as:

$$\begin{aligned} H(u_N^b, u_N^s, z_N) - H_2(u_N^b, u_N^s, z_N)(1 - \xi_N)(u_N^s - u_{N-1}^s) &= o[(1 - \xi_N)], \\ H(u_N^b, u_N^s, z_N) - H_1(u_N^b, u_N^s, z_N)(1 - \xi_N)(u_N^b - u_{N-1}^b) &= o[(1 - \xi_N)], \end{aligned}$$

where H_j is the partial derivative with respect to the j^{th} argument, and $o[(1 - \xi_N)]/(1 - \xi_N)$

converges to 0 as ξ_N converges to 1. Rearranging the terms and take limits, we obtain:

$$\lim_{\xi_N \rightarrow 1} H_2(u_N^b, u_N^s, z_N)(u_N^s - u_{N-1}^s) - H_1(u_N^b, u_N^s, z_N)(u_N^b - u_{N-1}^b) = \lim_{\xi_N \rightarrow 1} o[(1-\xi_N)]/(1-\xi_N) = 0. \quad (99)$$

This equation coincides with the FOC for (91). Hence, the solution to the alternating-offer round game corresponds to the Nash solution with disagreement points (u_{N-1}^b, u_{N-1}^s) .

Terminal payoffs

We now make the following proposition for the determination of the terminal payoffs starting from any arbitrary round, and let N be the total number of rounds. When solving the game with N rounds, we take the limit on the probability of negotiation breakdown. We solve the game by taking ξ_N to one first and obtain the solution to the subgame beginning from round N . Then we solve round $N - 1$, taking the limit of ξ_N at 1 as given. Then we take ξ_{N-1} to one, and so on.

We also need to expand the notation slightly. Let (u_{n-1}^b, u_{n-1}^s) be a given intermediate payoff at the beginning of round n , and let d_{n-1} be the corresponding cumulative transfer of assets; i.e., $H(u_{n-1}^b, u_{n-1}^s, d_{n-1}) = 0$. Define $F(u_{n-1}^b, u_{n-1}^s) = (u_n^b, u_n^s)$ to be the solution of

$$\max_{u_n^b, u_n^s} (u_n^b - u_{n-1}^b)(u_n^s - u_{n-1}^s) \quad \text{s.t.} \quad H(u_n^b, u_n^s, d_{n-1} + z/N) = 0. \quad (100)$$

Proposition 9 *Consider the subgame starting from the beginning of round $n \in \{1, \dots, N\}$ with intermediate payoffs, (u_{n-1}^b, u_{n-1}^s) , where $H(u_{n-1}^b, u_{n-1}^s, d_{n-1}) = 0$. Take limits in the following order: $\xi_N \rightarrow 1, \xi_{N-1} \rightarrow 1, \dots, \xi_n \rightarrow 1$. The terminal payoffs, (u_N^b, u_N^s) , are obtained recursively from (u_{n-1}^b, u_{n-1}^s) according to:*

$$\max_{u_{n+j}^b, u_{n+j}^s} (u_{n+j}^b - u_{n+j-1}^b)(u_{n+j}^s - u_{n+j-1}^s) \quad \text{s.t.} \quad H(u_{n+j}^b, u_{n+j}^s, z_{n+j}) = 0, \quad j = 0 \dots N - n, \quad (101)$$

where $z_{n+j} = d_{n-1} + (1 + j)z/N$.

The recursion (101) generates a sequence of payoffs, $\{(u_{n+j}^b, u_{n+j}^s)\}_{j=0}^{N-n}$, where each element, (u_{n+j}^b, u_{n+j}^s) , corresponds to the Nash solution of a bargaining problem with endogenous disagreement points, $(u_{n+j-1}^b, u_{n+j-1}^s)$, and Pareto frontier corresponding to the wealth z_{n+j} . We illustrate this construction in Figure 6 for the subgame starting in $N - 2$.

We prove the proposition by induction. We have shown that the proposition holds for round N . We now show that if the proposition holds for some arbitrary round n , then it holds for round $n - 1$. Consider the beginning of round $n - 1$ with intermediate payoffs, (u_{n-2}^b, u_{n-2}^s) , where $H(u_{n-2}^b, u_{n-2}^s, d_{n-2}) = 0$. We also assume that at round $n - 1$, it is the consumer to make the first offer.

In order to characterize the outcome of the alternating offer bargaining game in round $n - 1$ we need to compute the payoffs in case the negotiation ends without an agreement. In the event of a breakdown in round $n - 1$, then the players move to round n but keep the same intermediate payoffs, (u_{n-2}^b, u_{n-2}^s) . By inductive assumption, since the proposition holds for round n , the terminal payoffs in that subgame, denoted (u_{N-1}^b, u_{N-1}^s) , are given by $(u_{N-1}^b, u_{N-1}^s) = F^{N-n}(u_{n-1}^b, u_{n-1}^s)$, if we take the limits $\xi_N \rightarrow 1, \xi_{N-1} \rightarrow 1, \dots, \xi_n \rightarrow 1$, in that order.

Since our induction hypothesis allows us to compute the terminal payoffs from any intermediate payoffs in the beginning of round n , for any outcome from round $n - 1$, we can compute the continuation value. First let

$$\mathcal{H}_N = \{(u_N^b, u_N^s) \geq 0 : H(u_N^b, u_N^s, z_N) \geq 0\}$$

be the set of all possible individually rational final payoffs given the initial disagreement point. We use $\mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s)$ to denote the set of all terminal payoffs, (u_N^b, u_N^s) , attainable from (u_{n-2}^b, u_{n-2}^s) , for which the corresponding allocation is given by (y_{n-2}, d_{n-2}) , according to the induction hypothesis, if an offer at round $n - 1$ is accepted:

$$\begin{aligned} \mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s) &= \{F^{N-n+1}(u_{n-1}^b, u_{n-1}^s) : \exists(y_{n-1}, d_{n-1}) \geq (y_{n-2}, d_{n-2}), d_{n-1} - d_{n-2} \leq z/N \\ &\quad \text{such that } u_{n-1}^b = u(y_{n-1}) - d_{n-1}, u_{n-1}^s = -v(y_{n-1}) + d_{n-1}\}. \end{aligned}$$

Note that $\mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s) \subset \mathcal{H}_N$ is nonempty, as $(u_{N-1}^b, u_{N-1}^s) \equiv F^{N-n+1}(u_{n-2}^b, u_{n-2}^s) \in \mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s)$, which is attained if no trade is offered. Moreover, $(\hat{u}_N^b, \hat{u}_N^s) = F^{N-n+2}(u_{n-2}^b, u_{n-2}^s) \in \mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s)$ as well, which is attained if the offer corresponding to $(\hat{u}_{n-1}^b, \hat{u}_{n-1}^s) = F(u_{n-2}^b, u_{n-2}^s)$, denoted by $(\hat{y}_{n-1}, \hat{d}_{n-1})$, is offered and accepted. Moreover, since the cumulative offer, $(\hat{y}_{n-1}, \hat{d}_{n-1})$, is interior, i.e., $(\hat{y}_{n-1}, \hat{d}_{n-1}) > (y_{n-2}, d_{n-2})$, by continuity, there exists a neigh-

neighborhood \mathcal{O} around $(\hat{u}_N^b, \hat{u}_N^s)$ such that

$$\mathcal{O} \cap \mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s) \quad (102)$$

is open relative to \mathcal{H}_N .

Thus, using these terminal payoffs, the game in round $n - 1$ can be reduced to the following game: the two players take turns to make an offer $(u_N^b, u_N^s) \in \mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s)$. If accepted, the game ends with the terminal payoff (u_N^b, u_N^s) . Otherwise, with probability ξ_{n-1} the other player makes an offer; with probability $1 - \xi_{n-1}$ the game ends with payoff (u_{N-1}^b, u_{N-1}^s) . Note that only payoffs $(u_N^b, u_N^s) \geq (u_{N-1}^b, u_{N-1}^s)$ are relevant, for offers that lead to other payoffs are dominated by them. We claim that for ξ_{n-1} sufficiently large, the equilibrium payoffs, (u_N^b, u_N^s) , solve the following system of equations:

$$H[u_N^b, (1 - \xi_{n-1})u_{N-1}^s + \xi_{n-1}u_N^s, z_N] = 0, \quad (103)$$

$$H[(1 - \xi_{n-1})u_{N-1}^b + \xi_{n-1}u_N^b, u_N^s, z_N] = 0. \quad (104)$$

First we note that if $\mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s) = \mathcal{H}_N^{IR} \equiv \{(u_N^b, u_N^s) \in \mathcal{H}_N : (u_N^b, u_N^s) \geq (u_{N-1}^b, u_{N-1}^s)\}$, then this follows from the same argument as that for round N . The set \mathcal{H}_N^{IR} consists of all individually rational final payoffs relative to the disagreement point (u_{N-1}^b, u_{N-1}^s) . Now, since $\mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s) \subset \mathcal{H}_N$ and anything that is not individually rational is dominated by (u_{N-1}^b, u_{N-1}^s) , the proof is still valid as long as the final payoffs correspond to the solutions, $[u_N^b, (1 - \xi_{n-1})u_{N-1}^s + \xi_{n-1}u_N^s]$ and $[(1 - \xi_{n-1})u_{N-1}^b + \xi_{n-1}u_N^b, u_N^s]$, belong to $\mathcal{U}_N^b(u_{n-2}^b, u_{n-2}^s)$. By earlier argument we know that those solutions converge to $(\hat{u}_N^b, \hat{u}_N^s)$. Thus, for ξ_{n-1} sufficiently large, such solutions also belong to \mathcal{O} given by (102). Finally, the fact that the solution converges to the Nash solution as ξ_n approaches 1 follows exactly the same argument as round N . This proves that the proposition holds at $n - 1$. Given that it holds at N , by induction it holds for all $n \geq 0$.

Intermediate payoffs

We determine the equilibrium terminal payoffs at the start of the whole game by using the initial condition $(u_0^b, u_0^s) = (0, 0)$ and (101), i.e.,

$$\max_{u_n^b, u_n^s} (u_n^b - u_{n-1}^b) (u_n^s - u_{n-1}^s) \quad \text{s.t.} \quad H\left(u_n^b, u_n^s, \frac{n}{N}z\right) = 0.$$

We obtain a sequence $\{(u_n^b, u_n^s)\}_{n=0}^N$ where the last term corresponds to the terminal payoffs. Let's now denote $\{(\tilde{u}_n^b, \tilde{u}_n^s)\}_{n=0}^N$ the sequence of intermediate payoffs along the SPE. We determine this sequence by backward induction starting from $(\tilde{u}_N^b, \tilde{u}_N^s) = (u_N^b, u_N^s)$. Consider the alternating offer game in round N . Its solution is given by

$$(u_N^b, u_N^s) = \arg \max_{u_N^b, u_N^s} (u_N^b - \tilde{u}_{N-1}^b) (u_N^s - \tilde{u}_{N-1}^s) \quad \text{s.t.} \quad H(u_N^b, u_N^s, z) = 0.$$

By the definition of $\{(u_n^b, u_n^s)\}_{n=0}^N$ it follows that $(\tilde{u}_{N-1}^b, \tilde{u}_{N-1}^s) = (u_{N-1}^b, u_{N-1}^s)$.

Let's now move to round $N-1$. The disagreement point is $(\hat{u}_{N-1}^b, \hat{u}_{N-1}^s)$ solution to

$$(\hat{u}_{N-1}^b, \hat{u}_{N-1}^s) = \arg \max_{u_{N-1}^b, u_{N-1}^s} (u_{N-1}^b - \tilde{u}_{N-2}^b) (u_{N-1}^s - \tilde{u}_{N-2}^s) \quad \text{s.t.} \quad H\left(u_{N-1}^b, u_{N-1}^s, \frac{N-1}{N}z\right) = 0.$$

Given this disagreement point the terminal payoffs solve:

$$\max_{u_N^b, u_N^s} (u_N^b - \hat{u}_{N-1}^b) (u_N^s - \hat{u}_{N-1}^s) \quad \text{s.t.} \quad H(u_N^b, u_N^s, z) = 0.$$

It follows that $(\hat{u}_{N-1}^b, \hat{u}_{N-1}^s) = (u_{N-1}^b, u_{N-1}^s)$ and hence $(\tilde{u}_{N-2}^b, \tilde{u}_{N-2}^s) = (u_{N-2}^b, u_{N-2}^s)$. We can iterate this procedure and obtain that $(\tilde{u}_n^b, \tilde{u}_n^s) = (u_n^b, u_n^s)$ for all n . This then proves (23).

Gradual bargaining: limit as $N \rightarrow \infty$

The FOCs of the Nash problems above give

$$\frac{u_n^s - u_{n-1}^s}{u_n^b - u_{n-1}^b} = \frac{H_1\left(u_n^b, u_n^s, \frac{n}{N}z\right)}{H_2\left(u_n^b, u_n^s, \frac{n}{N}z\right)}.$$

Denote $\tau = nz/\delta N$. Divide both the numerator and the denominator of the left side by $z/\delta N$ and take the limit as N tends to infinity to obtain $u^{s'}(\tau)/u^{b'}(\tau)$. This gives:

$$\frac{u^{s'}(\tau)}{u^{b'}(\tau)} = \frac{H_1(u_\tau^b, u_\tau^s, \delta\tau)}{H_2(u_\tau^b, u_\tau^s, \delta\tau)}.$$

This differential equation coincides with (8).

Appendix D. Repeated Rubinstein game: the asymmetric case

We now generalize the game of Section 3 and study succinctly the case where consumers and producers are asymmetric by assuming that they bargain according to the generalized Nash solution in each of the $N \in \mathbb{N}$ stages of the game. The consumer's bargaining power is θ and the producer's bargaining power is $1 - \theta$. As before, one could provide strategic foundations for the use of the generalized Nash solution in each stage by considering a Rubinstein (1982) alternating-offer game where the risk of breakdown after an offer has been rejected depends on the identity of the responder.

The N -round game is solved by backward induction. Consider the last stage and suppose the interim agreement is $\tilde{o} \equiv (\tilde{y}, \tilde{p})$ with $\tilde{y} < y^*$ (so that there are gains from trade). The solution to the subgame with a single remaining stage, $o_1(\tilde{o}) \equiv (y_1, p_1)$, is

$$o_1(\tilde{o}) \in \arg \max_{y_1, p_1} [u(y_1) - p_1 - u(\tilde{y}) + \tilde{p}]^\theta [-v(y_1) + p_1 + v(\tilde{y}) - \tilde{p}]^{1-\theta} \quad \text{s.t. } p_1 - \tilde{p} \leq \frac{z}{N}. \quad (105)$$

The payoffs in case of disagreement correspond to \tilde{o} . The feasibility constraint requires that the consumer does not spend more than the last z/N units of assets on the bargaining table. We now move to stage $n = N - 1$ where we keep the same notation for the interim agreement, $\tilde{o} = (\tilde{y}, \tilde{p})$. The disagreement point is then $o_1(\tilde{o})$. The solution is the final

outcome, $o_2(\tilde{o}) \equiv (y_2, p_2)$, given by:

$$o_2(\tilde{o}) \in \arg \max_{y_2, p_2} [u(y_2) - p_2 - u(y_1) + p_1]^\theta [-v(y_2) + p_2 + v(y_1) - p_1]^{1-\theta} \quad \text{s.t.} \quad p_2 - \tilde{p} \leq \frac{2z}{N}. \quad (106)$$

The players who have perfect foresight negotiate the final outcome, o^2 , by taking into account that the agreement of stage $N - 1$ affects the outcome of the last stage as given by (105). The solution is obtained by applying the generalized Nash solution recursively. Given \tilde{o} , the disagreement point in stage $N - 1$, (y_1, p_1) , is obtained from (105). Given (y_1, p_1) , the offer (y_2, p_2) is obtained from (106). We need to show that there is an interim agreement in $N - 1$ that makes (y_2, p_2) feasible in round N . From the comparison of (105) and (106) it follows immediately that this interim agreement is $o_1(\tilde{o})$. So, ultimately, it is as if the solution in each stage is the naive general Nash solution with a backward-looking disagreement point.

We can iterate this reasoning to obtain a sequence of offers, $\{(y^n, p^n)\}_{n=0}^N$ with $(y^0, p^0) = (0, 0)$, that satisfies:

$$(y_n, p_n) \in \arg \max_{y, p} [u(y) - p - u(y_{n-1}) + p_{n-1}]^\theta [-v(y) + p + v(y_{n-1}) - p_{n-1}]^{1-\theta} \quad \text{s.t.} \quad p_n \leq \frac{nz}{N}. \quad (107)$$

As long as $p_n \leq \frac{nz}{N}$ binds, the solution takes the form:

$$\frac{z}{N} = \frac{(1 - \theta)v'(y_n) [u(y_n) - u(y_{n-1})] + \theta u'(y_n) [v(y_n) - v(y_{n-1})]}{\theta u'(y_n) + (1 - \theta)v'(y_n)}.$$

Summing across all stages, and assuming that $y_N < y^*$, the total output solves

$$z = \sum_{n=1}^N \int_{y_{n-1}}^{y_n} \frac{(1 - \theta)v'(y_n)u'(x) + \theta u'(y_n)v'(x)}{\theta u'(y_n) + (1 - \theta)v'(y_n)} dx.$$

The integrand is a weighted average of $u'(x)$ and $v'(x)$ where the weights depend on bargaining powers. As N goes to infinity, the right side converges to the asymmetric gradual solution described in Section 2.3.